Statistical Modeling by Wavelets

3 Wavelets

3.2 Discretization Of The Continuous Wavelet Transform

3.3 Multiresolution Analysis

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Discretization Of The Continuous Wavelet Transform

The critical sampling defined by $a = 2^{-j}, b = k\,2^{-j}, \; j,k \in Z$

For more general sampling, given by

$a = a_0^{-j}, b = k\,b_0\,a_0^{-j}, \; j,k \in Z, \quad a_0 > 1, b_0 > 0$

numerically stable reconstructions are possible if the system

$\{\psi_{jk}(x), j, k \in Z\}$ constitutes a frame.Here

$$\psi_{j,k}(x) = a_0^{j/2}\,\psi\left(\frac{x - k\,b_0 a_0^{-j}}{a_0^{-j}}\right) = a_0^{j/2}\psi(a_0^{j}x - kb_0)$$

$$\Phi(w)\; in\; L_2(R) \xrightarrow{FourierTransformation} \phi(x) \xrightarrow{checkorthogpnal} \int \phi(x)\phi(x-k)dx = \delta_k$$

$$\downarrow check\; orthonormal$$

$$\sum \left|\Phi(w+2\pi l)\right|^2 = 1$$

$$\phi(x) \lrcorner\; yes\; no \mapsto \phi(x) = F^{-1}\left[\frac{\Phi(w)}{\sqrt{\sum \left|\Phi(w+2\pi l)\right|^2}}\right]$$

$$\downarrow$$

$$m_1(w) = -e^{iw}\overline{m_0(w+\pi)} \longleftarrow m_0(w) = \frac{\Phi(w)}{\Phi(w/2)} \longleftarrow \Phi(w)$$

$$\downarrow$$

$$\Psi(w) = \Psi(\frac{w}{2})m_1(w) \xrightarrow{Fourier\;\; ransformation} \psi(x) \quad in \quad W_0$$

Multiresolution Analysis

A multiresolution analysis(MRA) is a sequence of closed subspaces $V_n, n \in Z$ in $L_2(R)$ such that they lie in a containment hierarchy

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \qquad (3.8)$$

$\bigcap_j V_j = \{0\}$, $\overline{\bigcup_j V_j} = L_2(R)$

The hierarchy (3.8) is constructed such that

(1) V-spaces are self-similar, $f(2^j x) \in V_j \quad iff \quad f(x) \in V_0$

(2) There exists a scaling function $\phi \in V_0$ whose integer-translates span the space V_0,

$$V_0 = \left\{ f \in L_2(R) \middle| f(x) = \sum_k c_k \phi(x-k) \right\}$$

and for which the set $\{\phi(\bullet - k),\ k \in Z\}$ is an orthonormal basis.

Scaling equation

Assume $\int \phi(x)\,dx \neq 0$

Since $V_0 \subset V_1$, the function $\phi(x) \in V_0$ can be represented as a linear combination of functions from V_1

ie. $$\phi(x) = \sum_{k \in Z} h_k \sqrt{2}\, \phi(2x-k) \qquad (3.10)$$

for some coefficients h_k $, k \in Z$

The(possibly infinite) vector $\underset{\sim}{h} = \{ h_n,\ n \in Z \}$ will be called a **wavelet filter**.

Note:

(1) $\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$ called Daubechies' convention

and $\left\{ \phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k), j \ \mathit{fixed}, k \in Z \right\}$ is a basis of V_j

(2) $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$ called Mallat's convention

and $\left\{ \phi_{jk}(x) = 2^{j/2}\phi(2^{j}x - k), j \ \mathit{fixed}, k \in Z \right\}$ is a basis of V_j

Define $m_0(w) = \frac{1}{\sqrt{2}} \sum_{k \in Z} h_k e^{-ikw} = \frac{1}{\sqrt{2}} H(w)$ (3.12)

is called **transfer function** and it describes the behavior of the associated filter $\underset{\sim}{h}$ in the Fourier domain.

The period is 2π and that the filter taps $\{h_n,\ n \in Z\}$ are the Fourier coefficients of the function $H(w) = \sqrt{2}\, m_0(w)$

Prove $\Phi(w) = m_0(\frac{w}{2})\,\Phi(\frac{w}{2}) = \prod_{n=1}^{\infty} m_0(\frac{w}{2^n})$

Pf :

$$\Phi(w) = \int_{-\infty}^{\infty} \phi(x)\, e^{-iwx}\, dx$$

$$= \sum_k \sqrt{2}\, h_k \int_{-\infty}^{\infty} \phi(2x-k)\, e^{-iwx}\, dx$$

$$= \sum_k \frac{h_k}{\sqrt{2}} e^{-ikw/2} \int_{-\infty}^{\infty} \phi(2x-k)\, e^{-i(2x-k)w/2}\, d(2x-k)$$

$$= \sum_k \frac{h_k}{\sqrt{2}} e^{-ikw/2}\, \Phi(\frac{w}{2})$$

$$= m_0(\frac{w}{2})\,\Phi(\frac{w}{2}) = \prod_{n=1}^{\infty} m_0(\frac{w}{2^n}) \qquad (\because \Phi(0)=1)$$

Normalization

$$\sum_{k \in Z} h_k = \sqrt{2} \tag{3.16}$$

$pf:$

$$\int \phi(x)\,dx = \sqrt{2}\sum_k h_k \int \phi(2x-k)\,dx$$

$$= \sqrt{2}\sum_k h_k \frac{1}{2}\int \phi(2x-k)\,d(2x-k)$$

$$= \frac{\sqrt{2}}{2}\sum_k h_k \int \phi(x)\,dx$$

$$\because \quad \int \phi(x)\,dx \neq 0$$

$$\therefore \quad \sum_k h_k = \sqrt{2}$$

Orthogonality: For any $l \in Z$,

$$\sum_k h_k h_{k-2l} = \delta_l$$

$$Pf: \quad \phi(x)\phi(x-l) = \sqrt{2}\sum_k h_k \phi(2x-k)\phi(x-l) \qquad (3.18)$$

$$= \sqrt{2}\sum_k h_k \phi(2x-k)\sqrt{2}\sum_m h_m \phi(2(x-l)-m)$$

By integrating the both sides in (3.18) we obtain

$$\delta_l = 2\sum_k h_k \left[\sum_m h_m \frac{1}{2} \int \phi(2x-k)\phi(2x-2l-m)\, d(2x) \right]$$

$$= \sum_k \sum_m h_k\, h_m\, \delta_{k,\ 2l+m} = \sum_k h_k\, h_{k-2l}$$

$$\because \delta_0 = \int \phi(x)\phi(x)dx = \langle \phi(x), \phi(x) \rangle = \|\phi(x)\|^2 = 1$$

$$\therefore If \quad l = 0, \quad then\, \delta_0 = \sum_k h_k{}^2 = 1$$

The fact that the system $\{\phi(\bullet - k), k \in Z\}$ constitutes an orthonormal basis for V_0 can be expressed in the Fourier domain in terms of either $\Phi(w)$ *or* $m_0(w)$

(a) In terms of $\Phi(w)$: $\sum_{l=-\infty}^{\infty} |\Phi(w+2\pi l)|^2 = 1$

Pf: Parseval property: $\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$

By the [PAR] property of the Fourier transformation and the 2π - periodicity of e^{iwk}

$$\delta_k = \int_R \phi(x)\,\overline{\phi(x-k)}\,dx$$

$$= \frac{1}{2\pi}\int_R \Phi(w)\,e^{iwx}\,\overline{\Phi(w)}\,e^{-iw(x-k)}dw$$

$$= \frac{1}{2\pi}\int_R \Phi(w)\overline{\Phi(w)}e^{iwk}\,dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{l=-\infty}^{\infty}\left|\,\Phi(w+2\pi l)\right|^2 e^{iwk}\,dw \qquad (3.21)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(w)e^{iwk}dw$$

$choose \quad k = 0$

$$\delta_0 = 1 = \frac{1}{2\pi}\int_0^{2\pi} f(w)dw$$

$$\Rightarrow f(w) = \sum_{l=-\infty}^{\infty}\left|\Phi(w+2\pi l)\right|^2 = 1$$

(b) In terms of m_0 :

$$|m_0(w)|^2 + |m_0(w+\pi)|^2 = 1 \qquad (3.23)$$

Pf: Since $\sum_{l=-\infty}^{\infty} |\Phi(2w+2l\pi)|^2 = 1$, then by $\Phi(w) = m_0(\frac{w}{2})\,\Phi(\frac{w}{2})$

$$\sum_{l=-\infty}^{\infty} |m_0(w+l\pi)|^2 \, |\Phi(w+l\pi)|^2 = 1 \qquad (3.24)$$

$$1 = \sum_{k=-\infty}^{\infty} |m_0(w+2k\pi)|^2 \, |\Phi(w+2k\pi)|^2$$

$$+ \sum_{k=-\infty}^{\infty} |m_0(w+(2k+1)\pi)|^2 \, |\Phi(w+(2k+1)\pi)|^2$$

since the period of $m_0(w)$ is 2π

$$1 = \left|m_0(w)\right|^2 \sum_{k=-\infty}^{\infty} \left|\Phi(w + 2k\pi)\right|^2$$

$$+ \left|m_0(w+\pi)\right|^2 \sum_{k=-\infty}^{\infty} \left|\Phi(w+\pi) + 2k\pi\right|^2$$

$$= \left|m_0(w)\right|^2 + \left|m_0(w+\pi)\right|^2$$

Remark 3.3.1

By identity (3.20),any set of independent functions spanning V_0 $\{\phi(x-k),\ k\in Z\}$can be orthogonalized in the Fourier domain. The orthonormal basis is generated by integer-shifts of the function

$$F^{-1}\left[\frac{\Phi(w)}{\sqrt{\sum_{l=-\infty}^{\infty}\left|\Phi(w+2\pi l)\right|^2}}\right]$$

This normalization in the Fourier domain is sometimes used in constructing wavelet bases

Remark 3.3.2

The system $\{\phi(\bullet - k), k \in Z\}$ is a frame for V_0

iff

$$A \leq \sum_{l=-\infty}^{\infty} \left|\Phi(w + 2l\pi)\right|^2 \leq B$$

Where A and B are frame bounds.

Remark 3.3.3

Conditions $|m_0(w)|^2 + |m_0(w+\pi)|^2 = 1$ and $\sum_{k=-\infty}^{\infty} |\Phi(2w+2k\pi)|^2 = 1$ are not equivalent. The first is a necessary and the second is a sufficient condition for orthogonality.

For example,

$\phi(x) = \frac{1}{3}\ \mathrm{I}(\ 0 \le x \le 3\ )$ has $m_0(w) = \dfrac{1+e^{-3iw}}{2}$

It satisfies $m_0(0) = 1$ and $\left|\dfrac{1+e^{-3iw}}{2}\right|^2 + \left|\dfrac{1-e^{-3iw}}{2}\right|^2 = 1$

Since $\Phi(w) = e^{-3iw/2}\ \dfrac{\sin 3w/2}{3w/2}$

$$\sum_{k=-\infty}^{\infty} \left|\Phi(2w + 2k\pi)\right|^2 = \frac{1}{3} + \frac{4}{9}\cos w + \frac{2}{9}\cos 2w \ne 1$$

3.3.1 Derivation of a Wavelet Function

Whenever a sequence of subspaces satisfies MRA properties, there exists (though *not unique*)an orthonormal basis for $L_2(R)$

$$\left\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), j\ , k \in Z\right\} \qquad (3.25)$$

such that $\left\{\psi_{jk}(x), j\ \ fixed, k \in Z\right\}$ is an orthonormal basis of the "difference space" $W_j = V_{j+!} - V_j$

The function $\psi(x) = \psi_{00}(x)$ is called a **wavelet function** or **informally the mother wavelet.**

Since $\psi(x) \in V_1$ (because of the containment $W_0 \subset V_1$), it can be represented as

$$\psi(x) = \sum_{k \in Z} g_k \sqrt{2}\, \phi(2x - k) \tag{3.26}$$

for some coefficients g_k , $k \in Z$

Define

$$m_1(w) = \frac{1}{\sqrt{2}} \sum_k g_k e^{-ikw} \tag{3.27}$$

By a derivation similar to that in (3.14),we obtain the Fourier counterpart of(3.26) ,

$$\Psi(w) = m_1(\frac{w}{2})\, \Phi(\frac{w}{2}) \tag{3.28}$$

Prove $\sum_{l=-\infty}^{\infty} \Psi(w+2l\pi)\overline{\Psi(w+2l\pi)} = 0$ (*)

Pf: $\because W_0 \text{ and } V_0 \text{ are orthogonal}, \phi(x) \in V_0, \varphi(x) \in V_1$

$$0 = \int \psi(x)\phi(x-k)dx$$

$$= \frac{1}{2\pi}\int \Psi(w)\overline{\Psi(w)}e^{iwk}dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{l=-\infty}^{\infty} \Psi(w+2l\pi)\overline{\Psi(w+2l\pi)}\, e^{iwk}\, dw$$

By repeating the Fourier series argument, as in (3.20)

Then we get (*)

$$(*) \quad \Leftrightarrow m_1(w)\overline{m_0(w)} + m_1(w+\pi)\overline{m_0(w+\pi)} = 0 \qquad (3.29)$$

From (3.29), we conclude that there exists a function $\lambda(w)$ such that

$$(m_1(w),\ m_1(w+\pi)) = \lambda(w)(\overline{m_0(w+\pi)}, \quad -\overline{m_0(w)}) \qquad (3.30)$$

By substituting $\xi = w + \pi$ and by using the 2π -periodicity of m_0, m_1

We conclude that $\lambda(w) = -\lambda(w+\pi)$ and $\lambda(w)$ is 2π -periodic

Note: Any function $\lambda(w)$ of the form $e^{\pm iw}S(2w)$, where S is an $L_2([0,2\pi])$

We choose $\lambda(w)$ such that

(a) $\lambda(w)$ is 2π-periodic

(b) $\lambda(w) = -\lambda(w+\pi)$

(c) $|\lambda(w)|^2 = 1$

Define $m_1(w) = -e^{-iw}\overline{m_0(w+\pi)}$ (3.32)

Since $|m_1(w)| = |m_0(w+\pi)|$,the orthogonality condition (3.23) can be

rewritten as $|\, m_0(w)\,|^2 + |\, m_1(w)\,|^2 = 1$

By comparing the definition of m_1 in (3.27) with

$$m_1(w) = -e^{-iw}\frac{1}{\sqrt{2}}\sum_k h_k\, e^{ik(w+\pi)}$$

$$= \frac{1}{\sqrt{2}}\sum_k (-1)^{1-k}\, h_k\, e^{-iw(1-k)}$$

$$= \frac{1}{\sqrt{2}}\sum_n (-1)^n\, h_{1-n}\, e^{-iwn}$$

$$m_1(w) = \frac{1}{\sqrt{2}}\sum_k g_k e^{-ikw} \qquad (3.27)$$

We relate g_n and h_n as

$$g_n = (-1)^n h_{1-n} \tag{3.34}$$

In signal processing, the relation (3.34) is known as the **quadrature mirror relation** and the filter $\underset{\sim}{h}$ and $\underset{\sim}{g}$ as **quadrature mirror filters.**

Remark 3.3.4

Choosing $\lambda(w) = e^{iw}$ leads to the rarely used high-pass filter

$$g_n = (-1)^{n-1} h_{-1-n}$$

It is sometimes convenient to define g_n as $g_n = (-1)^n h_{1-n-M}$

where M is a "shift constant" .Such re-indexing of $\underset{\sim}{g}$ affects

only the shift-location of the wavelet function.

Thanks