Some Extensions

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Regularity of Wavelets

- Let

$$M_k = \int x^k \phi(x)dx \text{ and } N_k = \int x^k \psi(x)dx$$

be the kth moments of scaling and wavelet functions

Thm.3.5.1

Let $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), j, k \in Z$ be an orthonormal system of functions in $L_2(R)$,

$$|\psi(x)| \leq \frac{C_1}{(1+|x|)^\alpha}, \alpha > N,$$

and $\psi \in C^{N-1}(R)$,where the derivatves $\psi^{(k)}(x)$ are bounded for $k \leq N-1$

then,ψ has N vanishing moment,

$$N_k = 0, 0 \leq k \leq N-1$$

Thm.3.5.1

if,in additon,

$$|\phi(x)| \leq \frac{C_2}{(1+|x|)^{\alpha}}, \alpha > N$$

then,the associated function $m_0(w)$ is necessarily of the form

$$m_0(w) = (\frac{1+e^{iw}}{2})^N \cdot L(w),$$

where L is a 2π-periodic,C^{N-1}-function

Definition 3.5.1

The multiresolution analysis (or, the scaling function) is said to be r-regular if, for any $\alpha \in Z$

$$|\psi^{(k)}(x)| \leq \frac{C}{(1+|x|)^{\alpha}}, for\ k = 0, 1, \ldots, r$$

- Assume that a wavelet function $\psi(x)$ has N vanishing moments,i.e.,

$$N_k = 0, k = 0, 1, \ldots, N-1$$

which implies

$$\frac{d^k \Psi(w)}{dw^k}|_{w=0}, k = 0, 1, \ldots, r$$

$$\Rightarrow m_1^{(k)}(w)|_{w=0} = m_1^{(k)}(0) = 0, k = 0, 1, \ldots, r$$

$$\Rightarrow m_0^{(k)}(w)|_{w=\pi} = m_0^{(k)}(\pi) = 0, k = 0, 1, \ldots, r$$

$$\Rightarrow \sum_{n \in Z} n^k g_n = \sum_{n \in Z} (-1)^n n^k h_n = 0, k = 0, 1, \ldots, r$$

Sobolev and Hölder regularity exponent

- Sobolev α_N^*:

$$\alpha_N^* = sup\beta,\ such\ that\ \int (1+|w|)^\beta |\Phi(w)| dw < \infty$$

- Hölder α_N: α_N is the exponent of the Hölder space C_N^α to which the scaling function ϕ belongs.
- Theorem 3.5.2

$$\lim_{N\to\infty} \frac{\alpha_N}{N} = 1 - \frac{log3}{2log2} \approx 0.2075$$

The Least Asymmetric Daubechies' Wavelets: Symmlets

- We referred to the Daubechies family of wavelets as the extremal phase family. As we pointed out before, they cannot be symmetric. However, it is possible to construct wavelets with compact support that are "more symmetric".

Approximations and Characterizations of Functional Spaces

- Any $L_2(R)$ function f can be represented as

$$f(x) = \sum_{j,k} d_{jk}\psi_{jk}(x)$$

- $L_2(R) = \bigoplus_{j=-\infty}^{\infty} W_j$, for any fixed j_0, $L_2(R) = V_{j_0} \oplus \bigoplus_{j=j_0}^{\infty} W_j$,thus

$$f(x) = \sum_{k} c_{j_0,k}\phi_{j_0,k}(x) + \sum_{j \geq j_0} \sum_{k} d_{jk}\psi_{j,k}(x)$$

The first sum in the above function is an orthogonal P_{j_0} of f on V_{j_0}.

- When both f and ϕ have n continuous derivatives, there exists a constant C such that

$$\|(I - P_{j_0})f\|_{L_2} \leq C \cdot 2^{-nj_0}\|f\|_{L_2}$$

- A range of important function spaces can be fully characterized by wavelets

- A function f belongs to the Hölder space C^s if and only if there is a constant C such that in an r-regular MRA $(r > s)$ the wavelet coefficients satisfy
  - $|c_{j_0,k}| \leq C$
  - $|d_{j,k}| \leq C \cdot 2^{-j(s+\frac{1}{2})}, j \geq j_0, k \in Z$
- A function f belongs to the Sobolev space $W_2^s(R)$ space if and only if

$$\sum_{j,k} |d_{jk}|^2 \cdot (1 + 2^{2js}) < \infty$$

- The function $f = \sum_k c_{j_0,k}\phi_{j_0,k}(x) + \sum_{j \geq j_0} \sum_k d_{jk}\psi_{j,k}(x)$ belongs to the $B^{\sigma}_{p,q}$ space if its wavelet coefficients satisfy
  - $$(\sum_k |c_{j0,k}|)^{1/p} \leq \infty$$
  - $$\{(\sum_{i \in I_j} 2^{j(\sigma+1/2-1/p)}|d_i|^p)^{1/p}, j \geq j_0\}$$

Daubechies-Lagarias Algorithm

Let ϕ be the function of the DAUBN wavelet. the support of ϕ is [0,2N-1]. Let $x \in (0,1)$, $dyad(x) = \{d_1, d_2, \ldots, d_n\}$ be the set of 0-1 digits in the dyadic representation of $x (x = \sum_{j=1}^{\infty} d_j 2^{-j})$
Let $\tilde{h} = (h_0, h_1, \ldots, h_{2N-1})$ be the wavelet filter coefficients. Define two $(2N-1) * (2N-1)$ matrices as:

$$T_0 = (\sqrt{2} \cdot h_{2i-j-1})_{1 \leq i,j \leq 2N-1}, T_1 = (\sqrt{2} \cdot h_{2i-j})_{1 \leq i,j \leq 2N-1}$$

Theorem 3.5.4

$$\lim_{n\to\infty} T_{d_1} \cdot T_{d_2} \cdots T_{d_n}$$

$$= \begin{pmatrix} \phi(x) & \phi(x) & \cdots & \phi(x) \\ \phi(x+1) & \phi(x+1) & \cdots & \phi(x+1) \\ \vdots & & & \\ \phi(x+2N-2) & \phi(x+2N-2) & \cdots & \phi(x+2N-2) \end{pmatrix}$$

Moment Condition

Assume that

$$N_k = \int_R x^k \psi(x) dx = 0, for\ k = 0, 1, \ldots, N-1$$

Thus we can get following relations involving coefficient of the filter $\tilde{h}$

- $\sum_{i=0}^{2N-1} h_i = \sqrt{2}$
- $\sum_{i=0}^{2N-1} (-1)^i i^k h_i = 0, k = 0, 1, \ldots, N-1$
- $\sum_{i=0}^{2N-1} h_i h_{i+2k} = \delta_k, k = 0, 1, \ldots, N-1$
- We obtained 2N+1 equations with 2N unknowns; however the system is solvable since the equations are not linearly independent.

Example 3.5.5

for N=2, we obtain the system:

$$\begin{cases} h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ -h_1 + h_2 - h_3 = 0 \\ h_0 h_2 + h_1 h_3 = 0 \end{cases}$$

which has the familiar solution
$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$

Interpolating (Cardinal) Wavelets

- Definition 3.5.2 A scaling function ϕ is interpolating if $\phi(n) = \delta_n$
- Definition 3.5.3 A scaling function is shift-interpolating if $\phi(n + \tau) = \delta_n$, for some constant τ
- $\because \sum_n (x - n)\phi(x - n) = M_1, \therefore \tau = M_1$

Example 3.5.4 DAUB2 wavelet is almost shift-interpolating

$$
\begin{aligned}
M_1 &= \frac{3 - \sqrt{3}}{2} \\
\phi(M_1) &= 1 \\
\phi(M_1 + 1) &\approx 0 \\
\phi(M_1 + 1) &\approx 0
\end{aligned}
$$

Pollen-Type Parameterization of Wavelets

Pollen parameterization for $N = 2$ (four-tap filters).[$s = 2\sqrt{2}$]

n	h_n for $N = 2$
0	$(1 + \cos\varphi - \sin\varphi)/s$
1	$(1 + \cos\varphi + \sin\varphi)/s$
2	$(1 - \cos\varphi + \sin\varphi)/s$
3	$(1 - \cos\varphi - \sin\varphi)/s$

for $\varphi = \frac{\pi}{4}, h_0 = \frac{\sqrt{2}}{4}, h_1 = \frac{2+\sqrt{2}}{4}, h_2 = \frac{\sqrt{2}}{4}, h_3 = \frac{-2+\sqrt{2}}{4}$

and Fig.3.18 obtained by point-to-point application of Daubechies-Lagarias Algorithm.