3.4 Some important wavelet bases

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＊Outline

3.4.1 Haar's Wavelets

3.4.2 Shannon's Wavelets

3.4.3 Meyer's Wavelets

3.4.4 Franklin's Wavelets

3.4.5 Daubechies' Compactly Supported Wavelets

∗3.4.1 Haar's Wavelets

Want to find: $\psi(x)$ in time-domain

First we known: $\phi(x) = 1(0 \le x \le 1)$

Step1: Find $m_0(\omega) = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}} e^{-i\omega 0}) + \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}} e^{-i\omega 1}) = \frac{1+e^{-i\omega}}{2}$.

From $m_0 = \frac{1}{\sqrt{2}} \sum_{k \in Z} h_k e^{-ik\omega}$.

∗3.4.1 Haar's Wavelets

$$\because \phi(x) = \phi(2x) + \phi(2x-1)$$

$$= \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x) + \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x-1),$$

$$\left(\phi(x) = \sum_{k\in Z} h_k \sqrt{2}\phi(2x-1)\right)$$

$\therefore$ *the wavelet filter coefficients*

$$h_0 = h_1 = \frac{1}{\sqrt{2}}.$$

Step2: Find $m_1(\omega) = \dfrac{1-e^{-i\omega}}{2}.$

∗3.4.1 Haar's Wavelets

From $m_1(\omega) = -e^{-i\omega}\overline{m_0(\omega+\pi)}.$

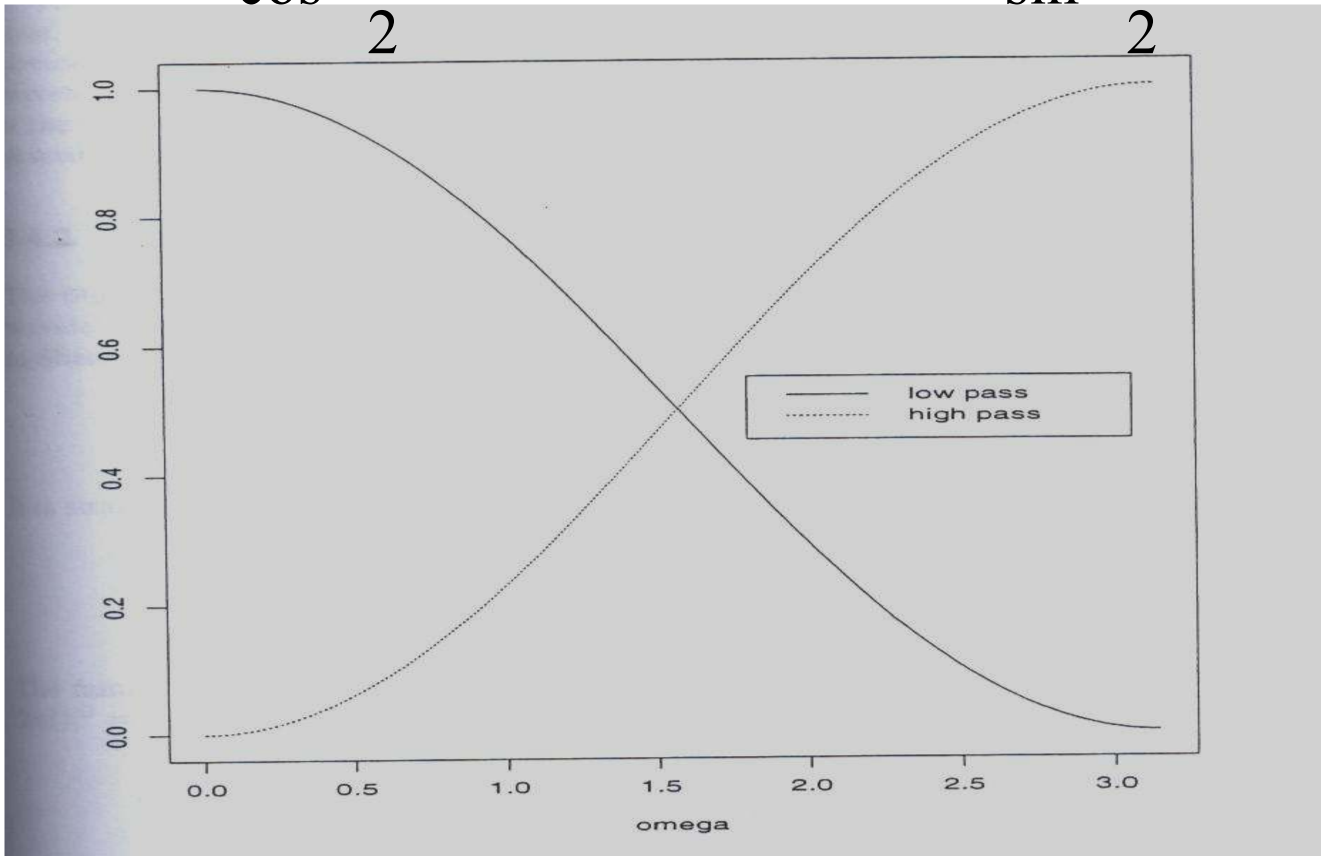
Step3: Check the orthogonality condition

$$\left|m_0(\omega)\right|^2 + \left|m_1(\omega)\right|^2 = 1$$

$$\Rightarrow \cos^2\frac{\omega}{2} + \sin^2\frac{\omega}{2} = 1$$

See Fig. 3.6

$$\cos^2\frac{\omega}{2} \qquad \sin^2\frac{\omega}{2}$$



∗3.4.1 Haar's Wavelets

Step4: Find the $\Psi(\omega)$ function in Fourier-domain

$$\Psi(\omega) = m_1\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) = \frac{1 - e^{-i\omega/2}}{2}\Phi\left(\frac{\omega}{2}\right)$$

Step5: Take the inverse Fourier transformation of $\Psi(\omega)$

$$\psi(x) = \phi(2x) - \phi(2x - 1)$$

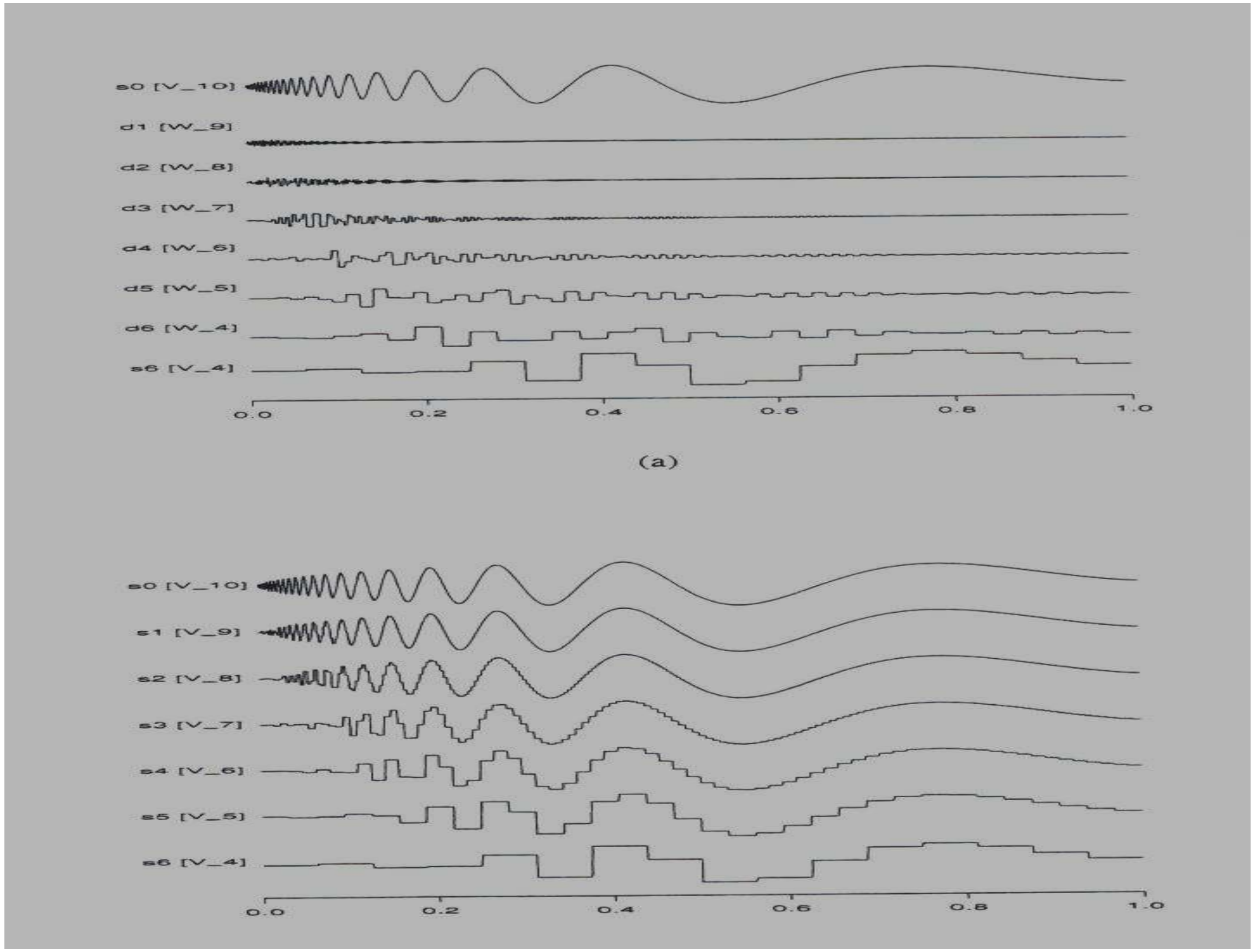
where $\psi(x)$ is called the Haar wavelet function

∗3.4.1 Haar's Wavelets

Some poor properties of Haar's Wavelet bases

1. Discontinuous function
2. Decay at the slow rate of $O\left(\frac{1}{n}\right)$ in the frequency domain.

see Fig. 3.7 Multiresolution analysis of the Doppler function



$*$3.4.2 Shannon's Wavelets

Want to find: $\psi(x)$ in time-domain

First we known: $\Phi(\omega) = 1(-\pi \le \omega \le \pi)$

Step1: Find the $\Phi(\omega)$ inverse Fourier transformation $\phi(x)$

$$\phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega x} d\omega = \frac{\sin(\pi x)}{\pi x}.$$

∗3.4.2 Shannon's Wavelets

Step2: Check the orthonormal condition

$$\sum_{-\infty}^{\infty} |\Phi(\omega + 2\pi l)|^2 = 1$$

Step3: Find the $m_0(\frac{\omega}{2})$

$$m_0(\frac{\omega}{2}) = \begin{cases} 1 & for\ \omega \in [-\pi, \pi] \\ 0 & for\ \omega \in [-2\pi, -\pi) \cup (\pi, 2\pi]. \end{cases}$$

$$\because \Phi(\omega) = m_0(\omega/2)\Phi(\omega/2)$$

Take $2\pi - periodicity$,

$$m_0(\omega) = \sum_{k \in Z} 1\left(-\frac{\pi}{2} + 2k\pi \le \omega \le \frac{\pi}{2} + 2k\pi\right)$$

∗3.4.2 Shannon's Wavelets

Step4: Find the $\Psi(\omega)$ function in Fourier-domain

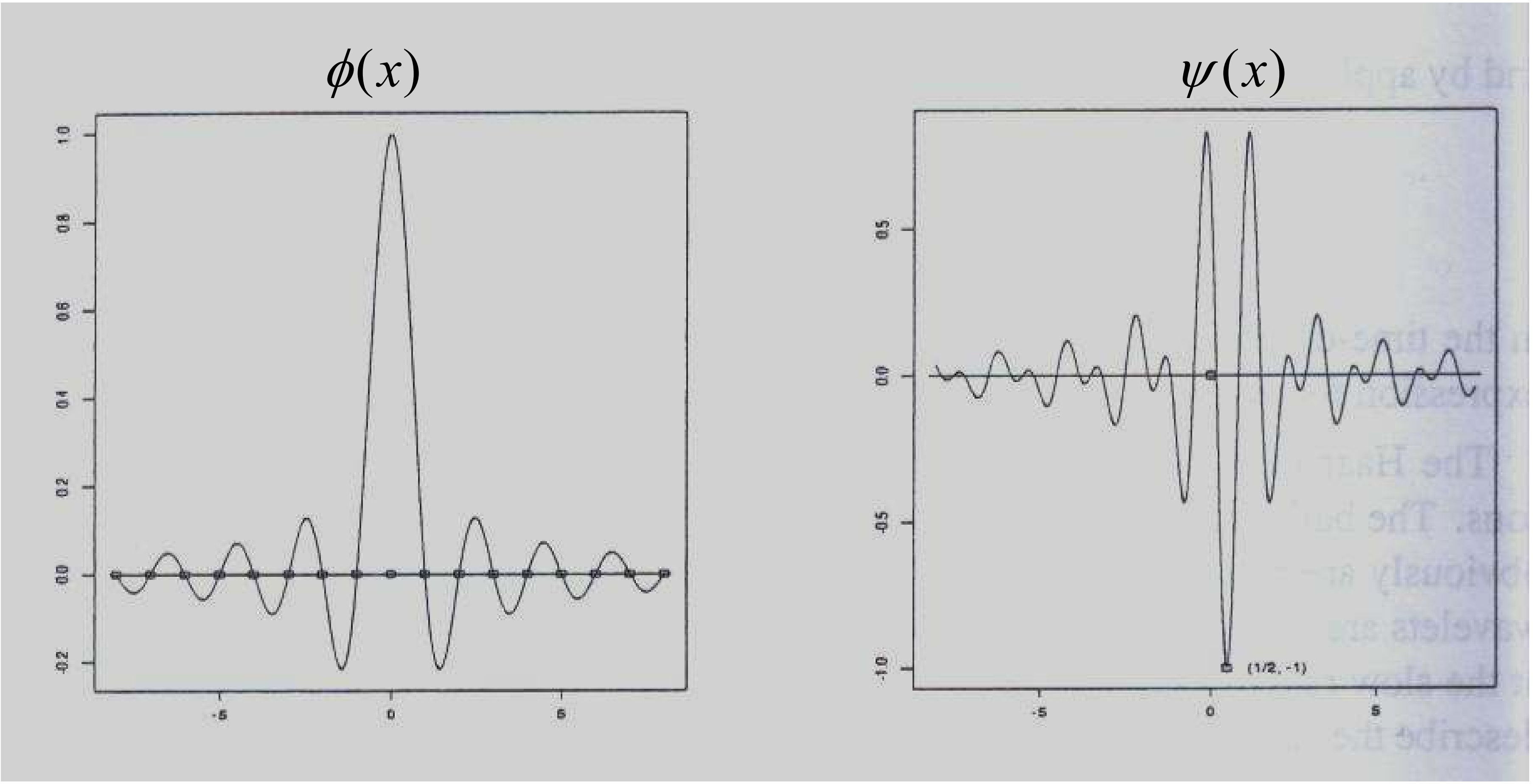
$$\Psi(\omega) = -e^{-i\omega/2} 1\left(\pi \le |\omega| \le 2\pi\right) = -e^{-i\omega/2}\left[\Phi\left(\frac{\omega}{2}\right) - \Phi(\omega)\right].$$

$$\because m_1(\omega) = -e^{-i\omega}\overline{m_0(\omega+\pi)} \quad \text{and} \quad \Psi(\omega) = m_1\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right)$$

Step5: Take the inverse Fourier transformation of $\Psi(\omega)$

$$\psi(x) = \phi(x - \frac{1}{2}) - 2\phi(2x-1)$$ ← Shannon's wavelet bases

See the Fig. 3.8.



∗3.4.2 Shannon's Wavelets

Step 6: Find the Shannon's filter

$$h_k = \frac{1}{2\pi}\int H(\omega)e^{i\omega x}d\omega = \ldots = \frac{1}{k\pi}\sin k\frac{\pi}{2}.$$

The Shannon's wavelet base is good in the frequency domain

but it has poor time localization properties.

∗3.4.3 Meyer's Wavelets

Ideas:

Meyer suggested modifying Shannon's $\Phi(\omega)$ function by smoothing sharp edges at $\omega = \pm\pi$, while preserving the orthogonality condition

$$\sum_{-\infty}^{\infty} |\Phi(\omega + 2\pi l)|^2 = 1$$

∗3.4.3 Meyer's Wavelets

First define the scaling function

$$\Phi(\omega)=\begin{cases}1 & |\omega|\le\frac{2\pi}{3}\\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega|-1\right)\right], & \frac{2\pi}{3}\le|\omega|\le\frac{4\pi}{3}\\ 0 & otherwise\end{cases}$$

where ν function satisfies: 1. $\nu(x)+\nu(1-x)=1$

2. $\nu(x)=\begin{cases}0 & x\le 0\\ 1 & x\ge 1.\end{cases}$

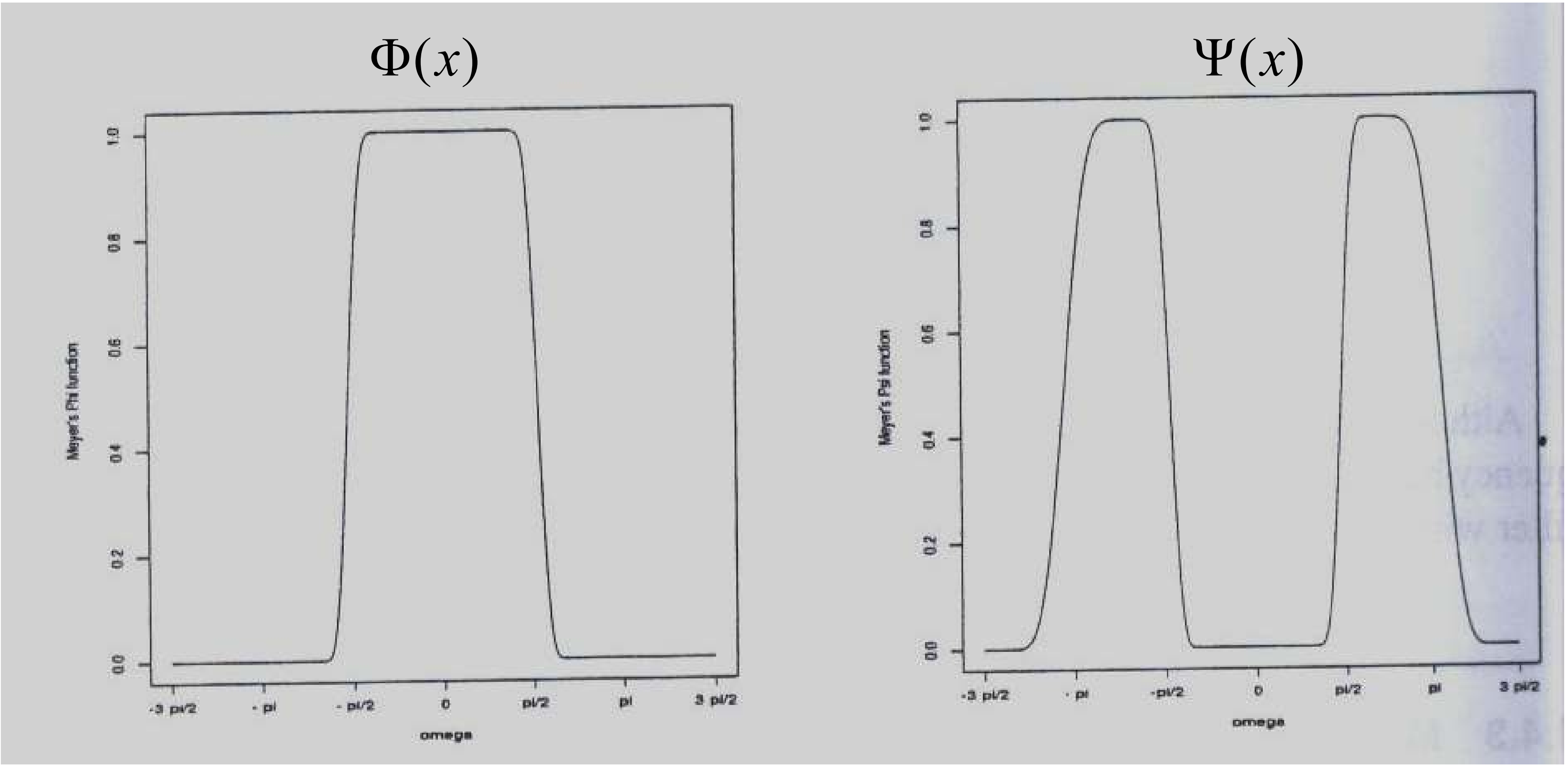
∗3.4.3 Meyer's Wavelets

Here for example: $\nu_\theta(x)$ be the random variable X's CDF, and $X \sim Beta(\theta,\theta)$

$$\nu_\theta(x) = \frac{1}{B(\theta,\theta)}\int_0^x t^{\theta-1}(1-t)^{\theta-1}dt$$

$B(\theta,\theta)$ is Euler's beta function.

See the Fig. 3.9 $\nu(x) = x^4(35 - 84x + 70x^2 - 20x^3)$.



$\ast$3.4.3 Meyer's Wavelets

Meyer's construction

Step1: $m_0\left(\frac{\omega}{2}\right)=\Phi(\omega) \quad -\pi\le\omega\le\pi.$

$\because$ *In the Fourier domain* $m_0\left(\frac{\omega}{2}\right)=\Phi(\omega)\Big/\Phi\left(\frac{\omega}{2}\right)$

then $\Phi\left(\frac{\omega}{2}\right)=1 \quad ,|\omega|\le\frac{4}{3}\pi$

take $2\pi-periodic \qquad m_0(\omega)=\sum_k \Phi[2(\omega+2k\pi)].$

∗3.4.3 Meyer's Wavelets

Step2: Demonstrate $\Phi(\omega) = m_0(\omega/2)\Phi(\omega/2)$

$(PF):$

$$m_0(\omega/2)\Phi(\omega/2) = \sum_k \Phi(\omega + 4k\pi)\Phi(\omega/2)$$
$$= \Phi(\omega)\Phi(\omega/2)$$
$$= \Phi(\omega)$$

Step3: Proving the orthogonality condition

$$\sum_k |\Phi(\omega + 2k\pi)|^2 = 1$$

∗3.4.3 Meyer's Wavelets

(PF):

$$
\begin{aligned}
&|\Phi(\omega)|^2+|\Phi(\omega-2k\pi)|^2\\
&=\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]+\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega-2\pi|-1\right)\right]\\
&=\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]+\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}(-\omega+2\pi)-1\right)\right]\\
&=\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]+\cos^2\left[\frac{\pi}{2}-\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\\
&=\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]+\sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]=1.
\end{aligned}
$$

Step4: Find an expression for the wavelet function in the Fourier domain

$$\Psi(\omega)$$

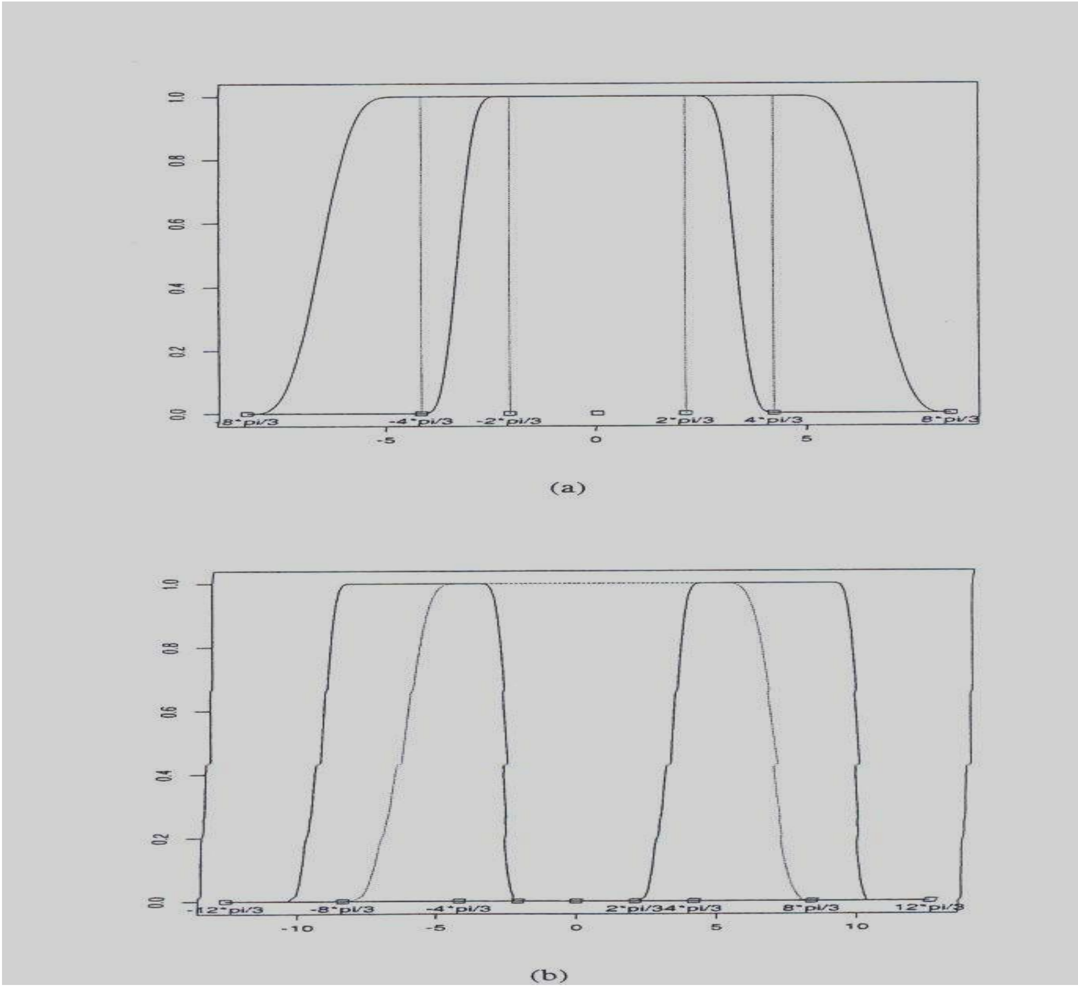
<Sol>:

$$\Psi(\omega) = -e^{-i\omega/2}\overline{m_0(\omega/2+\pi)}\Phi(\omega/2)$$

$$= -e^{-i\omega/2}\sum_{l}\Phi(\omega+2\pi(2l+1))\Phi(\omega/2)$$

$$= -e^{-i\omega/2}[\Phi(\omega-2\pi)+\Phi(\omega+2\pi)]\Phi(\omega/2)$$

See Fig 3.10.



-8*pi/3
-4*pi/3
-2*pi/3
2*pi/3
4*pi/3
8*pi/3
(a)
-12*pi/3
-8*pi/3
-4*pi/3
2*pi/34*pi/3
8*pi/3
12*pi/3
(b)

$$\Psi(\omega)=\begin{cases}-e^{-i\omega/2}\sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega|-1\right)\right], & \frac{2\pi}{3}\le|\omega|\le\frac{4\pi}{3}\\ -e^{-i\omega/2}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega|-1\right)\right], & \frac{4\pi}{3}\le|\omega|\le\frac{8\pi}{3}\\ 0, & otherwise\end{cases}$$

Step5: Find $\phi(x)$

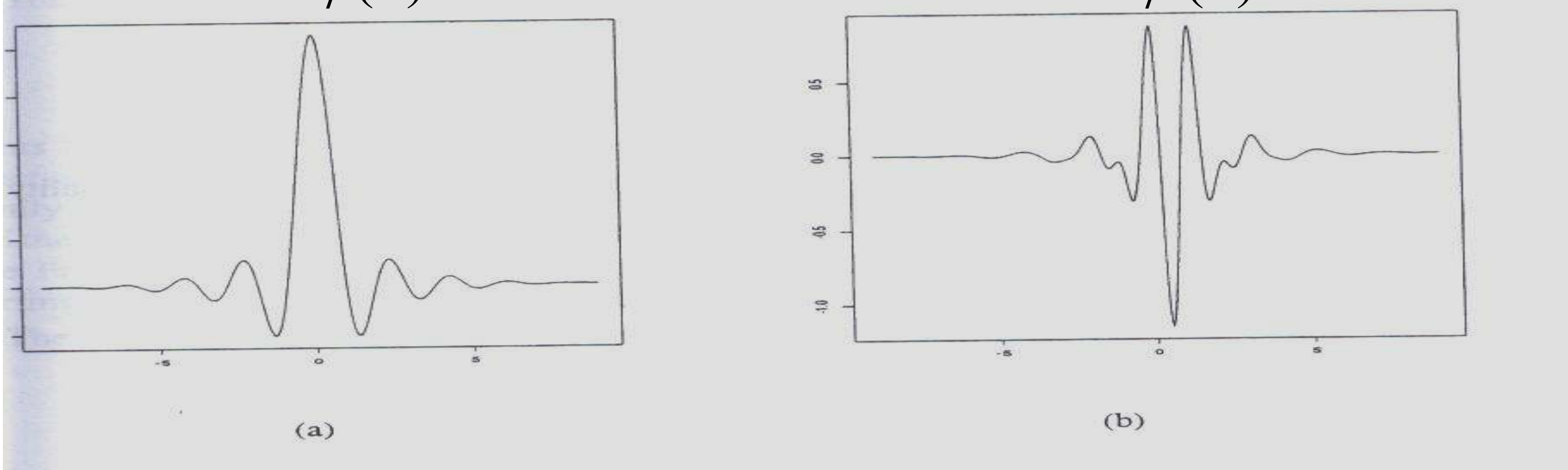
$$\phi(x)=\int\frac{1}{2}\Phi(\omega)e^{i\omega x}d\omega=\frac{1}{\pi}\int_{2\pi/3}^{4\pi/3}\Phi(\omega)\cos\omega x d\omega$$

Step6: Find $\psi(x)$

$$\psi(x) = -\frac{1}{\pi}\int_{2\pi/3}^{4\pi/3}\Phi(\omega/2)\Phi(\omega-2\pi\cos\omega(x-1/2)d\omega.$$

see Fig. 3.11.

$\phi(x)$ $\psi(x)$



Step7: Find h_n

$$h_n = \frac{\sqrt{2}}{\pi}\left(\int_0^{\pi/3} \cos n\omega \, d\omega + \int_{\pi/3}^{\pi/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{\pi}\omega - 1\right)\right] d\omega\right).$$

Remark 3.4.1

Walter introduced Meyer-type wavelets

Define the scaling function in the Fourier domain as

$$\Phi(\omega) = \left(\int_{\omega-\pi}^{\omega+\pi} dP\right)^{\frac{1}{2}} \quad \text{and} \quad \Psi(\omega) = e^{-i\omega/2}\left(\int_{|\omega|/2-\pi}^{|\omega|/2+\pi} dP\right)^{\frac{1}{2}}$$

where P is any probability measure supported on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

The orthogonality of scaling function verifies readily,

∗3.4.4 Franklin's Wavelets

The function $\Delta(x)$ is a "tent" function

$$\Delta(x) = (1-|x-1|)\mathrm{I}(0 \le x \le 2) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x < 2 \\ 0 & otherwise \end{cases}$$

$\{\Delta(x-k), \quad k \in Z\}$ span the space V_0

This tent function come from "Schauder basis" (page 3)

$\ast$3.4.4 Franklin's Wavelets

$\because \{\Delta(x-k), \quad k \in Z\}$ *are not orthogonal scaling function*

but According to (3.22) in Remark 3.31, it is possible to orthogonalize the

$\{\Delta(x-k), \quad k \in Z\}$ family

∗3.4.4 Franklin's Wavelets

◎ To orthogonalize the system $\{\Delta(x-k),\quad k\in Z\}$

$$\Delta(x)\xrightarrow{\text{Fourier transformation}}\hat{\Delta}(\omega)=e^{-i\omega}\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2$$

then find $S(\omega)=\sum_l|\hat{\Delta}(\omega+2l\pi)|^2=\sum_l\left(\frac{\sin\left(\frac{\omega}{2}+l\pi\right)}{\frac{\omega}{2}+l\pi}\right)^4$

by Lemma 3.4.1

$$S(\omega) = 1 - \frac{2}{3}\sin^2\frac{\omega}{2}$$

$\therefore$ The orthogonalized system is

$$\Phi(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2 [S(\omega)]^{-\frac{1}{2}}$$

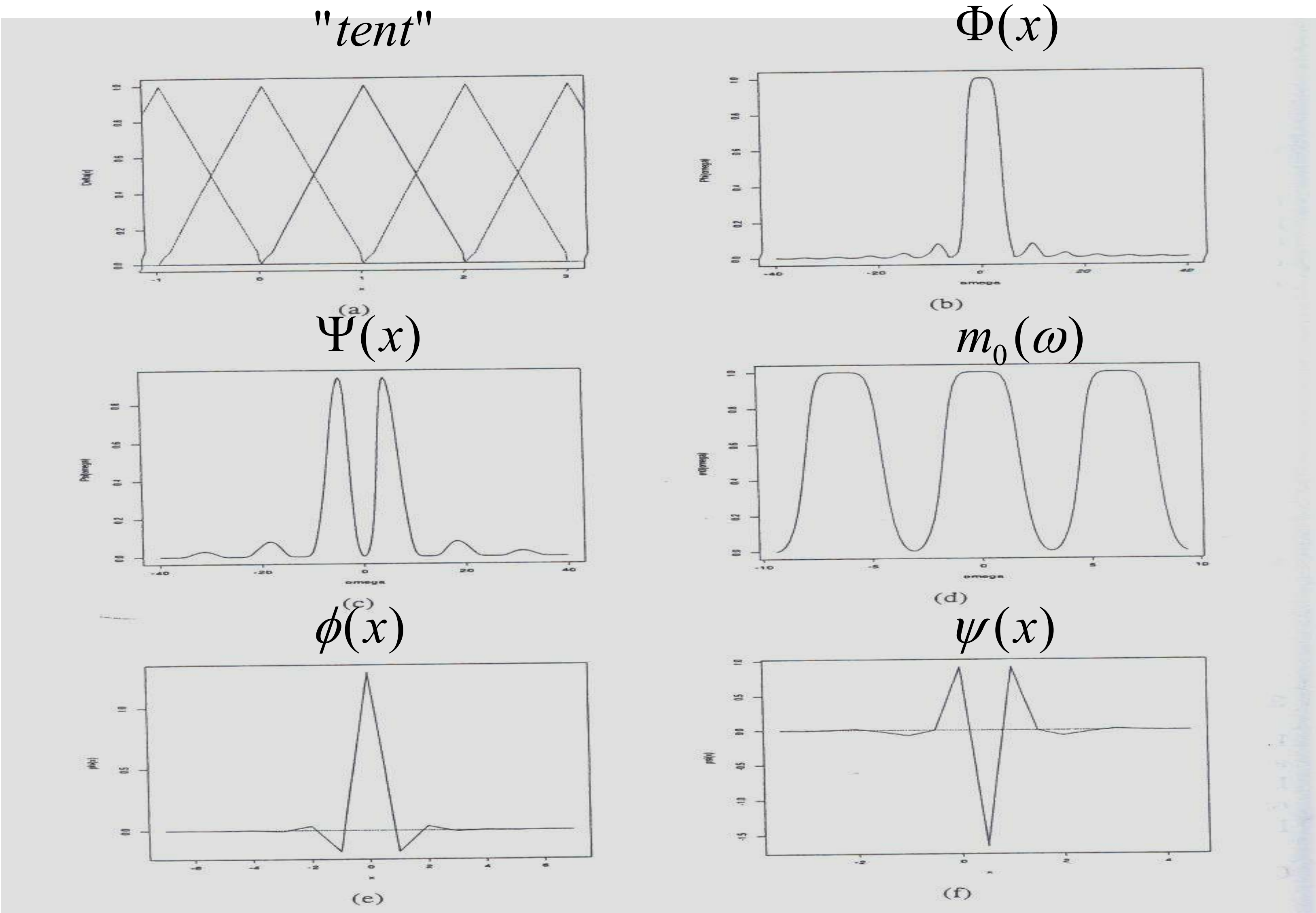
∗3.4.4 Franklin's Wavelets

$$\therefore m_0(\omega) = \frac{\Phi(2\omega)}{\Phi(\omega)} = \cos^2\frac{\omega}{2}\sqrt{\frac{2+\cos\omega}{2+\cos 2\omega}}.$$

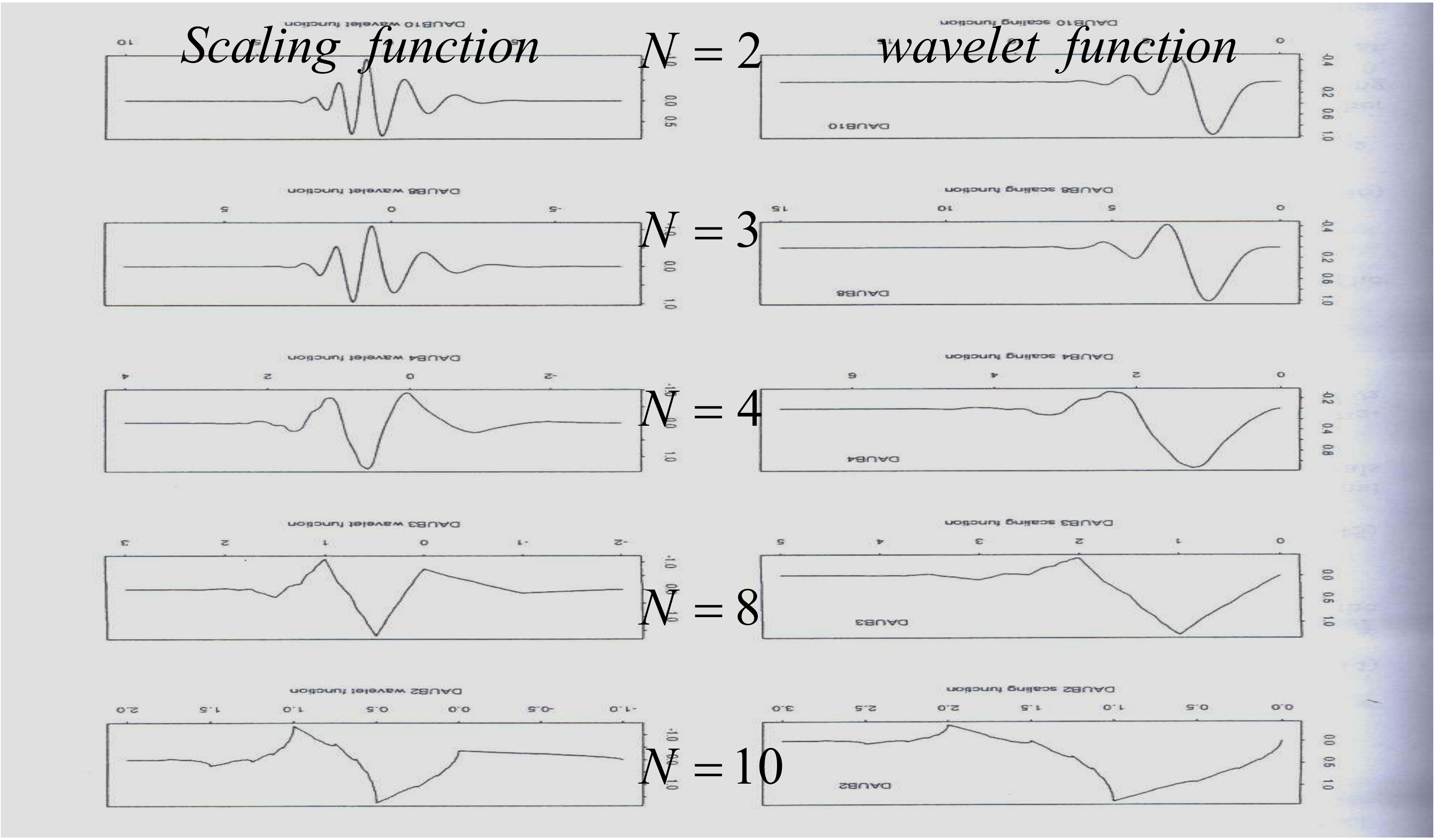
$$m_1(\omega) = -e^{-i\omega}\overline{m_0(\omega+\pi)}$$

$$\Psi(2\omega) = -e^{-i\omega}\overline{m_0(\omega+\pi)}\Phi(\omega).$$

$$\psi(x) = \psi(1-x)$$


"*tent*"
$\Phi(x)$
(a)
(b)
$\Psi(x)$
$m_0(\omega)$
(c)
(d)
$\phi(x)$
$\psi(x)$
(e)
(f)

∗ Daubechies’ Compactly Supported Wavelets



$\ast$ Daubechies' Compactly Supported Wavelets

1. By Theorem 3.5.1

$m_0(\omega)$ has the form $m_0(\omega) = \left(\dfrac{1+e^{-i\omega}}{2} \right)^N L(\omega),$

where $L(\omega)$ is a trigonometric polynomial.

∗ Daubechies' Compactly Supported Wavelets

2. The orthogonality condition $|m(\omega)|^2 + |m(\omega+\pi)|^2 = 1$ becomes

$$M_0(\omega) + M_0(\omega+\pi) = 1$$

where $M_0(\omega) = |m_0(\omega)|^2 = \left(\cos^2\dfrac{\omega}{2}\right)^N |L(\omega)|^2,$

∗ Daubechies' Compactly Supported Wavelets

3. Re-expressed as orthogonality condition in $y = \sin^2 \dfrac{\omega}{2}$

$$(1-y)^N P(y) + y^N P(1-y) = 1 \qquad (3.45)$$

4. By Bezout's Lemma, we can find the unique solution of (3.45) is

$$\sum_{k=0}^{N-1} \binom{N+k-1}{k} y^k$$

◎ 滿足 orthogonality condition 的 $|L(\omega)|^2$ 已經找到，並可推到 $|m_0(\omega)|^2$

$*$ Daubechies' Compactly Supported Wavelets

5. Find $|L(\omega)|^2$'s square root by Lemma 3.4.3 (Riesz)

Step1:

First let auxiliary polynomial P_A, such that $|L(e^{-i\omega})|^2 = |P_A(e^{-i\omega})|$

Step2:

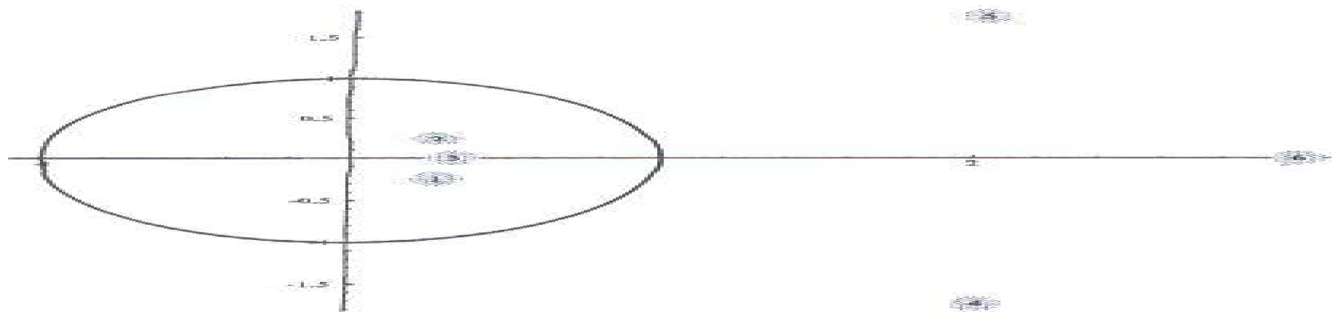
by Riesz Lemma, $P_A(z) = \frac{1}{2}\sum_{k=1-N}^{N-1} a_{|k|}\, z^{N-1+k}$.

＊ Daubechies' Compactly Supported Wavelets

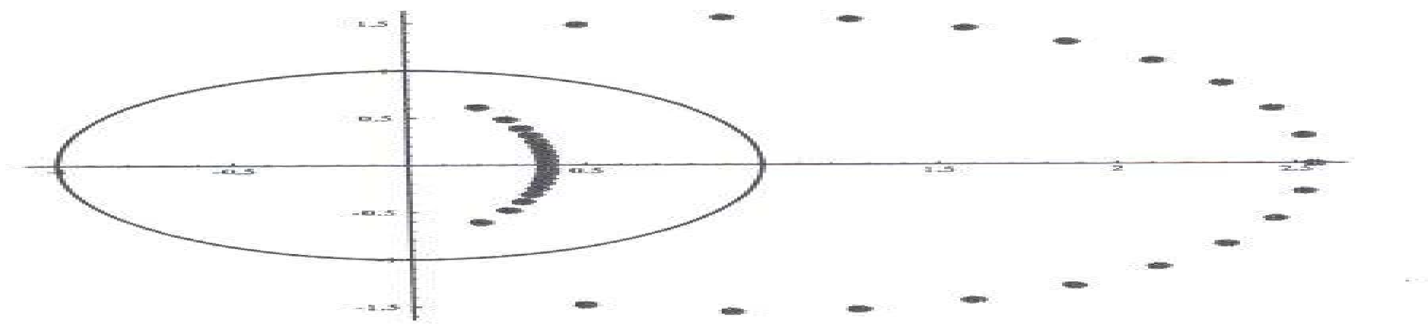
Step3:

Find the zeroes of $P_A(z)$ as $(r_i, \frac{1}{r_i}, z_i, \overline{z_i}, z_i^{-1}, \overline{z_i}^{-1})$

see the Fig.3.14 give locations of zeroes of $P_A(z)$ for $N = 4$ [Panel (a)]



(a)



(b)

Step4:

Factorized polynomial P_A as

$$P_A(z) = \frac{1}{2} a_{N-1} \left[\prod_{i=1}^{I} (z - r_i)(z - \frac{1}{r_i}) \right] \left[\prod_{i=1}^{I} (z - z_i)(z - \overline{z_i})(z - z_i^{-1})(z - \overline{z_i}^{-1}) \right].$$

Step5:

Replace $|(z - z_j)(z - \overline{z_j}^{-1})|$ by $|z_j|^{-1} |z - z_j|^2$ then $|P_A(z)|$ becomes

$$\frac{1}{2} |a_{N-1}| \prod_{i=1}^{I} |r_i^{-1}| \prod_{j=1}^{I} |z_j|^{-2} \left| \prod_{i=1}^{I} (z - r_i) \prod_{j=1}^{J} (z - z_j)(z - \overline{z_j}) \right|^2 .$$

Step6:

Because $|P_A(z)|^{\frac{1}{2}} = L(\omega)$, then $L(\omega)$ as follow

$$\pm(\frac{1}{2}|a_{N-1}|\prod_{i=1}^{I}|r_i^{-1}|\prod_{j=1}^{I}|z_j|^{-2})^{\frac{1}{2}}|\prod_{i=1}^{I}(z-r_i)\prod_{j=1}^{J}(z-z_j)(z-\overline{z_j})|.$$

Final:

We can find $m_0(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N L(\omega)$, then Find the coefficients

$h_0, h_1, \ldots, h_{2N-1}$ in the polynomial $\sqrt{2}m_0(\omega)$ are the desired wavelet

filter coefficients.

Thank you!