Wavelets

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Introduction

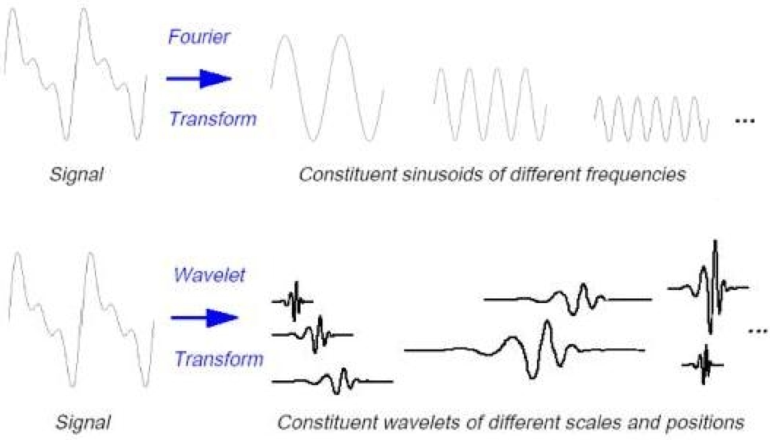

Figure 1: Fourier transform and Wavelet transform

Example

- Consider the following function

$$f(x) = -\frac{16}{9}x^2 + \frac{8}{3}x$$

$$f_j(x) = \begin{cases} V_{j0} & \text{, if } x \in [0, \frac{1}{z^j}), \\ V_{j1} & \text{, if } x \in [\frac{1}{z^j}, \frac{2}{2^j}), \\ \vdots & \\ V_{j,2^j-1} & \text{, if } x \in [1 - \frac{1}{2^j}, 1), \\ 0 & \text{, if } x = 1. \end{cases}$$

where

$V_{jk} = \int_{A_k} f(x)dx, \; A_k = [\frac{k}{2^j}, \frac{k+1}{2^j}), \; k = 0, 1, 2, \cdots, 2^j - 1.$

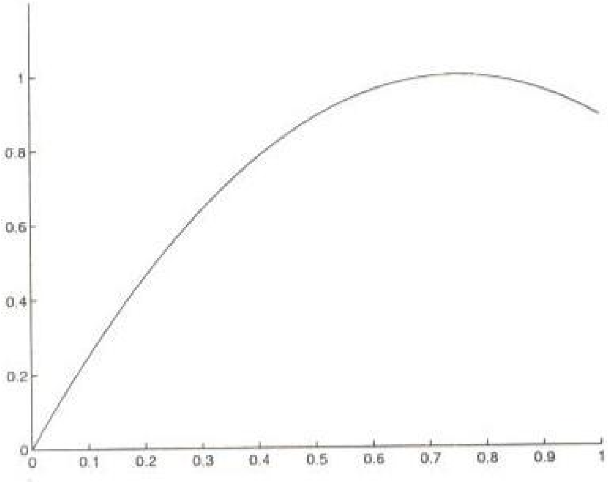

Figure 2: $f(x) = -\frac{16}{9}x^2 + \frac{8}{3}x$

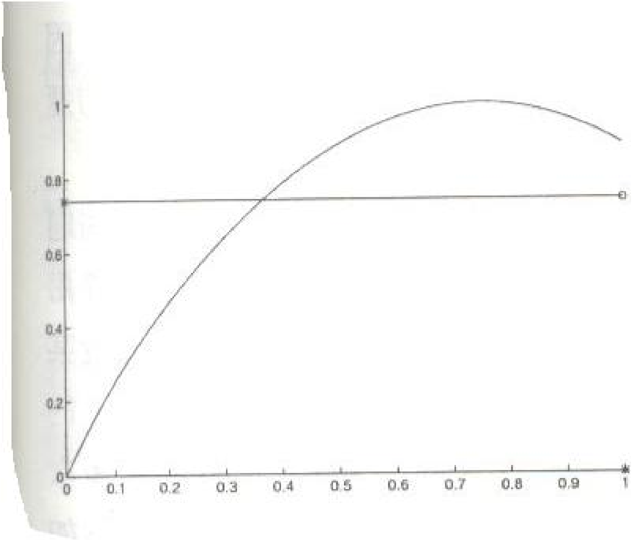

Figure 3: $f(x)$ and $f_0(x)$

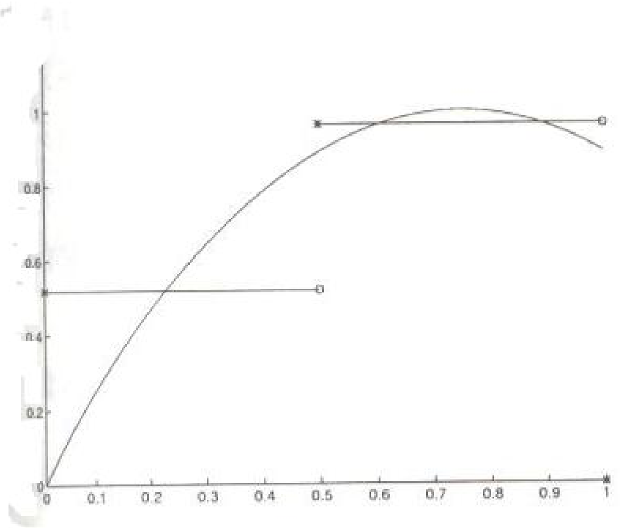

Figure 4: $f(x)$ and $f_1(x)$

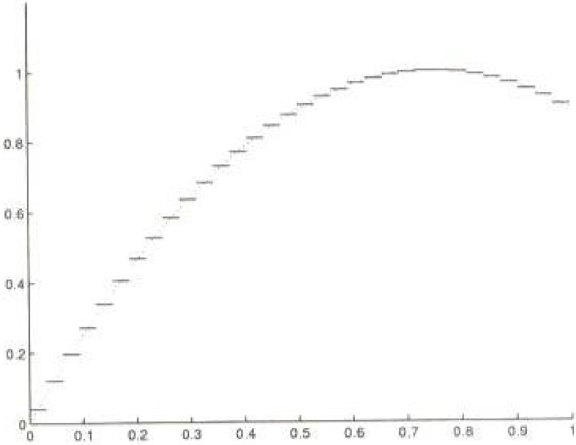

Figure 5: $f(x)$ and $f_5(x)$

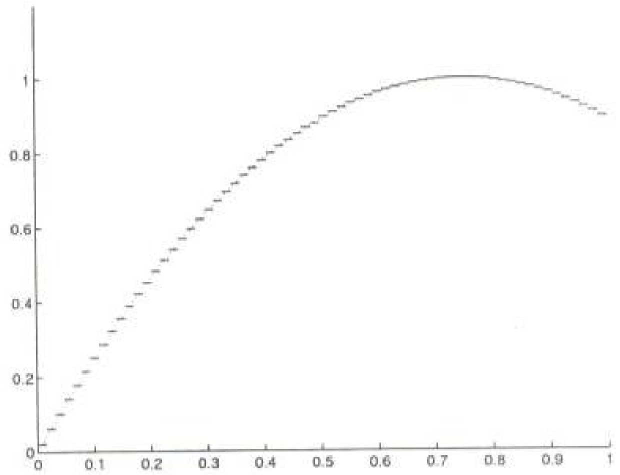

Figure 6: $f(x)$ and $f_6(x)$

Continuous Wavelet Transformation

- Statistics is mainly interested in discrete wavelet transformation.
- The research in both probability and time series are formulated in terms of continuous wavelet transformations.
- Let $\psi_{a,b}(x), a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$ be be family of functions defined as translation and re-scales of a single function $\psi(x) \in \mathbb{L}_2(\mathbb{R})$.

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi(\frac{x-b}{a}).$$

  called the wavelet function or the mother wavelet.
- Wavelet functions are usually normalized to $\|\psi_{a,b}(x)\| = 1$

- The admissibility condition

$$\mathcal{C}_\psi = \int_{\mathbb{R}} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty.$$

where $\Psi(\omega)$ is the Fourier transformation of $\psi(x)$.

- The admissibility condition implies $0 = \Psi(0) = \int \psi(x)dx$, hence it is called a "wavlet".
- If $\int \psi(x)dx = 0$, and $\int (1 + |x|^\alpha)|\psi(x)|dx < \infty$, then $\mathcal{C}_\psi < \infty$
- For any $\mathbb{L}_2$ function, the wavelet continuous wavelet transformation is

$$\mathcal{CWT}_f(a,b) = \langle f, \psi_{a,b} \rangle = \int f(x)\overline{\psi_{a,b}}dx,$$

where $(a,b) \in (\mathbb{R} \setminus \{0\}, \mathbb{R})$

Basic Properties

- Resolution of indentity

$$f(x) = \frac{1}{\mathcal{C}_\psi} \int_{\mathbb{R}^2} \mathcal{CWT}_f(a,b)\psi_{a,b}(x)\frac{dadb}{a^2},$$

if a is restricted to $\mathbb{R}^+$, then it be interpreted as a reciprocal of frequency. Hence

$$\int_0^\infty \frac{|\Psi(\omega)|^2}{\omega} d\omega = \frac{1}{2}\mathcal{C}_\psi < \infty$$

, so

$$f(x) = \frac{2}{\mathcal{C}_\psi} \int_{-\infty}^\infty \int_0^\infty \mathcal{CWT}_f(a,b)\psi_{a,b}(x)\frac{dadb}{a^2},$$

proof

$\forall f, g \in L_2(\mathbb{R})$. we have

$$\int_{-\infty}^{\infty} [CWT_f(a,b) \cdot \overline{CWT_g(a,b)}] db$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} \{ \int_{-\infty}^{\infty} f(t) \overline{\psi(\frac{t-b}{a})} dt \cdot \int_{-\infty}^{\infty} \overline{g(s)} \psi(\frac{s-b}{b}) ds \} db$$

$$= \frac{a^2}{|a|} \int_{-\infty}^{\infty} \{ \frac{1}{2\pi} \overline{\int_{-\infty}^{\infty} \overline{F(w)} e^{-ibw} dw} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G(w)} e^{-ibw} dw \} db.$$

$$= \frac{a^2}{2\pi|a|} [\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\widehat{\overline{F(b)}}} \cdot \widehat{\overline{G(b)}} db]$$

$$= \frac{a^2}{2\pi|a|} \int_{-\infty}^{\infty} F(x) \overline{G(x)} dx$$

by $\Psi_{a,b}(w) = a \cdot e^{-ibw} \cdot \Psi(aw)$, and

$< f(x), g(x) > = \frac{1}{2\pi} < f(\hat{w}), g(\hat{w}) >$

and,

$$\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}[CWT_f(a,b)\cdot\overline{CWT_g(a,b)}]db\}\frac{da}{a^2}$$

$$=\int_{-\infty}^{\infty}\{\frac{a^2}{2\pi|a|}\int_{-\infty}^{\infty}F(x)\overline{G(x)}dx\}\frac{da}{a^2}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\widehat{f(w)}\overline{\widehat{g(w)}}\cdot\int_{-\infty}^{\infty}\frac{|\Psi(aw)|^2}{a}da\}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\widehat{f(w)}\overline{\widehat{g(w)}}\cdot\int_{-\infty}^{\infty}\frac{|\Psi(y)|^2}{y}dy\}dw$$

$$=C_\psi\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f(w)}\overline{\widehat{g(w)}}dw$$

$$=C_\psi\frac{1}{2\pi}<\hat{f},\hat{g}>$$

$$=C_\psi<f,g>$$

hence,

$$
\begin{aligned}
& C_\psi \int f(x)\overline{g(x)}dx \\
&= \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}[CWT_f(a,b)\cdot\overline{CWT_g(a,b)}]db\}\frac{da}{a^2} \\
&= \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}[CWT_f(a,b)\cdot\overline{\int g(x)\overline{\psi_{a,b}(x)}dx}]db\}\frac{da}{a^2} \\
&= \int[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} CWT_f(a,b)\overline{g(x)}\psi_{a,b}(x)\frac{dadb}{a^2}]dx
\end{aligned}
$$

$\therefore f(x) = \frac{1}{C_\psi}\int_{\mathbb{R}^2} CWT_f(a,b)\psi_{a,b}(x)\frac{dadb}{a^2}$.

- Shifting Property: If $f(x)$ has a continuous wavelet transformation $\mathcal{CWT}_f(a,b)$ and $g(x) = f(x-\beta)$, then $\mathcal{CWT}_g(a,b) = \mathcal{CWT}_f(a,b-\beta)$

$$\begin{aligned}
\because \quad \mathcal{CWT}_g(a,b) &= \int g(x)\overline{\psi_{a,b}(x)}dx \\
&= \int f(x-\beta)\overline{\psi_{a,b}(x)}dx \\
&= \int g(y)\overline{\psi_{a,b-\beta}(x)}dy \\
&= \mathcal{CWT}_f(a,b-\beta)
\end{aligned}$$

- Scaling Property: If $g(x) = \frac{1}{\sqrt{s}}f(\frac{x}{s})$, then $\mathcal{CWT}_g(a,b) = \mathcal{CWT}_f(\frac{a}{s},\frac{b}{s})$.

- Energy Conservation:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{\mathcal{C}_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathcal{CWT}_f(a,b)|^2 \frac{dadb}{a^2}.$$

$$\because \; < f, f > = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} CWT_f(a,b) \cdot \overline{CWT_f(a,b)} \frac{dadb}{a^2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |CWT_f(a,b)|^2 \frac{dadb}{a^2}$$

- Localization: Let $f(x) = \delta(x - x_0)$ be the Dirac pulse at the point x_0, then

$$
\begin{aligned}
\mathcal{CWT}_f(a,b) &= \int \delta(x - x_0) \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}) dx \\
&= \int_{x_0}^{x_0+t} \lim_{t \to 0} \frac{1}{t} \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}) dx \\
&= \lim_{t \to 0} \frac{1}{t} \int_{x_0}^{x_0+t} \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}) dx \\
&= \lim_{t \to 0} \frac{\int_{x_0}^{x_0+t} \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}) dx}{t} \\
&= \frac{1}{\sqrt{a}} \psi(\frac{x_0 - b}{a})
\end{aligned}
$$

- Reproducing Kernel Property: Define $\mathbb{K}(u,v;a,b) = \langle \psi_{u,v}, \psi_{a,b} \rangle$, a continuous wavelet transformation of $f(x)$ is

$$
\begin{aligned}
F(u,v) =& \int f(x)\overline{\psi_{u,v}(x)}dx \\
=& \int \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_0^{\infty} CWT_f(a,b)\psi_{a,b}(x)\frac{dadb}{a^2}\overline{\psi_{u,v}(x)}dx \\
=& C_\psi \int_{-\infty}^{\infty} \int_0^{\infty} CWT_f(a,b) \int \psi_{a,b}(x)\overline{\psi_{u,v}(x)}dx \frac{dadb}{a^2} \\
=& \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_0^{\infty} \mathbb{K}(u,v;a,b)F(a,b)\frac{dadb}{a^2}.
\end{aligned}
$$

- Characterization of Regularity: Let $\int (1+|x|)|\psi(x)|dx < \infty$, and $\Psi(0) = 0$, if $f \in \mathbb{C}^\alpha$ (Hölder space), then $|\mathcal{CWT}_f(a,b)| \leq C|a|^{\alpha+1/2}$.

Example

- Continuous Haar transformation: Let

$$\psi_{a,b}^{HAAR}(x) = \frac{1}{\sqrt{a}}[1(b \leq x < \frac{a}{2} + b) - 1(\frac{a}{2} + b \leq x \leq a + b)],$$

$$a \in \mathbb{R}^+, b \in \mathbb{R}.$$

And let $F' = f$, then

$$\mathcal{CWT}_f(a,b) = \langle f, \psi_{a,b}^{HAAR} \rangle = \frac{2}{\sqrt{a}}[F(\frac{a}{2}+b) - \frac{F(b) + F(a+b)}{2}].$$

- Theorem 3.1.2: Let $\xi(x)$ and $\xi^{(n)}(x)$, $n \geq 1$, be $\mathbb{L}_2(\mathbb{R})$ functions and let $\xi^{(n)}(x) \neq 0$. Then $\psi(x) = \xi^{(n)}(x)$ is wavelet.
- Proof:

$$
\begin{aligned}
\mathcal{C}_\psi &= \int \frac{|\Psi(\omega)|^2}{|\omega|} d\omega \\
&= \int_{|\omega|\leq 1} |\omega|^{2n-1} |\hat{\xi}(\omega)|^2 d\omega + \int_{|\omega|<1} \frac{|\omega|^{2n} |\hat{\xi}(\omega)|^2}{|\omega|} d\omega \\
&\quad (\because \Psi(\omega) = \widehat{\xi^{(n)}(\omega)} = (i\omega)^n \widehat{\xi(\omega)}) \\
&\leq \int_{|\omega|\leq 1} |\hat{\xi}(\omega)|^2 d\omega + \int_{|\omega|<1} |\omega^n \hat{\xi}(\omega)|^2 d\omega \\
&\quad (\because \|\hat{\xi}(\omega)\|_\infty < \|\xi\| \Rightarrow \|\hat{\xi}(\omega)\|_\infty^2 < \|\xi\|^2) \\
&\leq \|\xi(x)\|_{\mathbb{L}_2}^2 + \|\xi^{(n)}\|_{\mathbb{L}_2}^2 < \infty.
\end{aligned}
$$

Discretization of the continuous wavelet transformation

- The critical sampling defined by $a = 2^{-j},\ b = k2^{-j},\ j,\ k\ \in \mathbb{Z}$
- For more general sampling, given by

$$a = a_0^{-j},\ b = kb_0a_0^{-j},\ j,\ k\ \in \mathbb{Z},\ a_0 > 1,\ b_0 > 0$$

and,

$$\psi_{jk}(x) = a_0^{j/2}\psi(\frac{x - kb_0a_0^{-j}}{a_0^{-j}}) = a_0^{j/2}\psi(a_0^{-j}x - kb_0)$$

THANKS