Some Important Function Spaces & Fundamentals of Signal Processing

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2.5 Some Important Function spaces

- $\mathbb{C}^n(\mathbb{R})$ A function belonges to the space $\mathbb{C}^n(\mathbb{R})$ if it is n-times continuously differentiable.

Sobolve Space

- $\mathbb{W}_2^s(\mathbb{R})$ Sobolve space is define by

$$\mathbb{W}_2^s(\mathbb{R}) = \{f| \ \|(1+|w|^2)^{s/2}\hat{f}(w)\|_{\mathbb{L}_2} < \infty\}$$

For s=0, this is just $\mathbb{L}_2(\mathbb{R})$

For s=1,2,3,. . . , the space $\mathbb{W}_2^s(\mathbb{R})$ consists of all $\mathbb{L}_2(\mathbb{R})$ functions that are s times differentiable and whose sth derivative belongs to $\mathbb{L}_2(\mathbb{R})$.

For s=-1,-2,-3,. . . , the space $\mathbb{W}_2^s(\mathbb{R})$ contains all distributions with a point support of order$\leq s$, $(\delta, \delta', \ldots, \delta^{(s-1)})$

Sobolve Space

For $f, g \in \mathbb{W}_2^s(\mathbb{R})$, the inner product is defined by

$$\langle f, g \rangle_s = \frac{1}{2\pi} \int \hat{f}(w)\overline{\hat{g}(w)}(1 + |w|^2)^s dw$$

The space $\mathbb{W}_2^s(\mathbb{R})$ is complete w.r.t. this inner product and thus is a Hilbert space.

Hölder Space

$\mathbb{C}^s(\mathbb{R})$ Hölder space is defined by

(i) $0 < s < 1, \mathbb{C}^s(\mathbb{R}) = \{f \in \mathbb{L}_\infty(\mathbb{R}) : \sup_h |\frac{f(x+h)-f(x)}{|h|^s}| < \infty\}$

(ii) $s = n + s', 0 < s' < 1,$
$\mathbb{C}^s(\mathbb{R}) = \{f \in \mathbb{L}_\infty(\mathbb{R}) \bigcap \mathbb{C}^n(\mathbb{R}) | \frac{d^n}{dx^n} f \in \mathbb{C}^{s'}(\mathbb{R})\}$

Besov space

$\mathbb{B}^{\sigma}_{pq}(I)$ The Besov space is defined by

- the class of all function f with a finite Besov norm.
- i.e. The Besov norm $\|f\|_{\mathbb{B}^{\sigma}_{p,q}} = \|f\|_{\mathbb{L}_p(I)} + |f|_{\mathbb{B}^{\sigma}_{p,q}} < \infty$

$|f|_{\mathbb{B}^{\sigma}_{p,q}}$ is the Besov seminorm.

Besov space

For selected

$\sigma > 0, 0 < p \leq \infty,$ and $0 < q \leq \infty$ chose r so that $r - 1 \leq \sigma \leq r$

The Besov seminorm of index (σ, p, q) is the defined by

- if $1 \leq q < \infty$, $|f|_{\mathbb{B}^{\sigma}_{p,q}} = [\int_0^{\infty} (h^{-\sigma} w_{r,p}(f;h))^q \frac{dh}{h}]^{(1/q)}$
- if $q = \infty$, $|f|_{\mathbb{B}^{\sigma}_{p,q}} = \text{sup}_h \, h^{-\sigma} w_{r,p}(f;h)$

the $w_{r,p}(f;h)$ is the rth modulus of smoothness of $f \in \mathbb{L}_p(I)$

Besov space

<u>Def</u>

$w_{r,p}(f;t) = \sup_{|h|\leq t} \|\triangle_h^{(r)} f\|_{\mathbb{L}_p(I_{rh})}$ with the supremum norm when $p = \infty$

- Let $\triangle_h^{(0)} f(t) = f(t)$ and
- $\triangle_h^{(r)} = \triangle_h^{(r-1)} f(x+h) - \triangle_h^{(r-1)} f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(t+kh)$
- $\triangle_h^{(r)}$ is defined for $x \in I_{rh} = \{x \in I | x + rh \in I\}$

The Besov spaces are very general space comprising most other spaces as special cases.

- Sobolev space $\mathbb{W}_2^s$ is $\mathbb{B}_{2,2}^s$
- Hölder space $\mathbb{C}^s$ is $\mathbb{B}_{\infty,\infty}^s$.
- The function f belongs to $\mathbb{B}_{p,q}^\sigma$, if there exist $f_0, g_0, g_1, g_2, \ldots, \in$ Sobolev space $\mathbb{W}_p^m$ and a sequence $\{\epsilon_0, \epsilon_1, \epsilon_2, \ldots\} \in l^q$ such that

$$\begin{aligned} f &= f_0 + g_0 + g_1 + g_2 + \ldots \in \mathbb{L}_p \\ \|g_j\|_p &\leq \epsilon_j 2^{-\sigma j}, j = 0, 1, 2, \ldots \\ \|g_j^{(m)}\|_p &\leq C\epsilon_j 2^{(m-\sigma)j}, j = 0, 1, 2, \ldots \end{aligned}$$

2.6 Fundamentals of Signal Processing

A filter H is a linear operator which maps l_2 to l_2.

For $x \in l_2$ the equation $y = Hx$ has a component-wise representation.

$$\begin{aligned} y(n) &= (h \star x)(n) = \sum_k h(k)x(n-k) \\ \text{where } h(k) &= h_k, k \in \mathbb{Z} \text{ are filter coefficients.} \end{aligned}$$

The filter coefficient may be obtained when the filter H is applied to the sequence $\tilde{u} = (\ldots, 0, 0, 1, 0, 0, \ldots)$(unit impulse at zero)

$$\tilde{h} = H\tilde{u} = (\ldots, h_0, h_1, \ldots)$$

Components h_i of $\tilde{h}$ are called "taps" of the filter.

- A filter band is a set of two or more filters.
- The filter is called "causal" when negative indice are not allowed in $\tilde{h}$.
- If the number of nonzero taps is finite, the filter is called *finite impulse response*(FIR), otherwise the filter is *infinite impulse response*(IIR).
- The function

$$H(w) = \sum_n h(n) e^{-inw}$$

is the frquency response function.

When the input is $x(n) = e^{inw}$ and filter taps are $(h_0, h_1, \ldots)$

$$\begin{aligned} y(n) &= \sum_k h(k)x(n-k) \\ &= h_0 e^{inw} + h_1 e^{i(n-1)w} + \ldots \\ &= (h_0 + h_1 e^{-iw} + \ldots)e^{inw} \\ &= H(w)x(n) \end{aligned}$$

$H(w)$ is a comples function,and can be representation in the form

$$H(w) = |H(w)|e^{-i\Phi(w)}$$

where $|H(w)|$ is the magnitude of $H(w)$ and $\Phi(w)$ is the phase angle.

- When $H(0) = 1$ and $H(\pi) = 0$, the filter is low-pass.
- When $H(0) = 0$ and $H(\pi) = 1$, the filter is high-pass.