Statistical Modeling by Wavelets

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2.3.1 Basic Properties

2.3.2 Poisson Summation Formula and Sampling Theorem

2.3.3 Fourier Series

2.3.4 Discrete Fourier Transform

Definition 2.3.1

The Fourier transformation of a function $f \in L_1(R)$ is defined by

$$\hat{f}(\omega) = F[f(x)] = \left\langle f(x), e^{i\omega x} \right\rangle = \int_R f(x)\overline{e^{i\omega x}}dx = \int_R f(x)e^{-i\omega x}dx.$$

If $\hat{f} \in L_1(R)$ is the Fourier transformation of $f \in L_1(R)$, then

$$f(\omega) = F^{-1}[\hat{f}(x)] = \frac{1}{2\pi}\int_R \hat{f}(x)e^{i\omega x}dx,$$ at every continuity point of f.

Detail

1) The function $\hat{f}(\omega)$ is, in general, a complex function of the form

$$\hat{f}(\omega) = |\hat{f}(\omega)| e^{i\varphi(\omega)}$$

2) The part $|\hat{f}(\omega)|$ is called the magnitude spectrum, and the part $\varphi(\omega)$ is called the phase spectrum.

3) If $f(x)$ is real, then

i) $\hat{f}(-\omega) = \overline{\hat{f}(\omega)}$, and

ii) $|\hat{f}(\omega)|$ is an even function and $\varphi(\omega)$ is an odd function of ω

Example 2.3.1 Let

$$f_l(x) = \begin{cases} 1, & |x| \leq l/2 \\ 0, & |x| > l/2. \end{cases}$$

Then, by taking into account the representation $\sin z = \dfrac{e^z - e^{-z}}{2i}$ we get

$$\hat{f_l}(\omega) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{-i\omega x} dx = -\frac{1}{i\omega} e^{-i\omega x} \Big|_{-l/2}^{l/2} = \frac{1}{\omega} \frac{e^{i\omega l/2} - e^{-i\omega l/2}}{i} = l \frac{\sin \omega \frac{l}{2}}{\omega \frac{l}{2}}.$$

2.3.1 Basic Properties

[BOU] Boundedness. $f \in L_\infty(R),\ \| \hat{f} \|_\infty \le \| f \|_1$.

[UC] Uniform Continuity. $\hat{f}(\omega)$ is uniformly continuous on $-\infty < \omega < \infty.$

[DEC] Decay. For $f \in L_1,\ \ \hat{f}(\omega) \to 0,$ when $|\omega| \to \infty,$

[LIN] Linearity. $F[\alpha f(x) + \beta g(x)] = \alpha F[f(x)] + \beta F[g(x)].$

[DER] Derivative. $F[f^{(n)}(x)] = (i\omega)^n \hat{f}(\omega).$

Properties

[PLA] Plancherel's Identity. $\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$; If $g = f$ one obtains

Plancherel's identity: $\| f \|^2 = \frac{1}{2\pi} \| \hat{f} \|^2$.

[SHI] Shifting. $F[f(x - x_0)] = e^{-i\omega x_0} \hat{f}(\omega)$.

[SCA] Scaling. $F[f(ax)] = \frac{1}{|a|} \hat{f}(\frac{\omega}{a})$.

[SYM] Symmetry. $F[F[f(x)]] = 2\pi f(-x)$.

Properties

[CON] Convolution. The convolution of f and g is defined as

$f*g(x)=\int f(x-t)g(t)dt$. One of the most important properties of Fourier transformations is $F[f*g(x)]=\hat{f}(\omega)\hat{g}(\omega)$.

[MOD] Modulation Theorem. $f(x)g(x)=\frac{1}{2\pi}F(\omega)*G(\omega)$.

[MOM] Moment Theorem

$$\int_R x^n f(x)dx=(i)^n \left.\frac{d^n \hat{f}(\omega)}{d\omega^n}\right|_{\omega=0}.$$

Example 2.3.2 Find $\hat{g}(\omega)$ for $f(x)\cos\omega_0 x$.

$\langle SOL \rangle$:

$\because$ $\cos\omega_0 = (e^{i\omega_0 x} + e^{-i\omega_0 x})/2$, then $f(x)\cos\omega_0 x$ equal the following equation

$f(x)\cos\omega_0 x = \frac{1}{2} f(x)e^{i\omega_0 x} + \frac{1}{2} f(x)e^{-i\omega_0 x}$, use [LIN] and [SHI] we can find

$$\hat{g}(\omega) = \frac{1}{2}\hat{f}(\omega - \omega_0) + \frac{1}{2}\hat{f}(\omega + \omega_0).$$

Theorem 2.3.1 (Poisson theorem) If function f is smooth and decays fast

$$\sum_{n=-\infty}^{\infty} f(x-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{2\pi k}{T}\right)e^{i2\pi kx/T}.$$

For $T=1$ and $x=0$

$$\sum_{n=-\infty}^{\infty} f(-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k).$$

Example 2.3.4

$$\sum_{n=-\infty}^{\infty} e^{-a|n|} = \sum_{n=-\infty}^{\infty} \frac{2a}{a^2 + (2n\pi)^2}, \; a > 0,$$

First we find the Fourier transformation of $e^{-a|x|}$ as follow

$$\text{let } f(x) = e^{-a|x|} \Rightarrow \hat{f}(x) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx$$

$$= \int_{-\infty}^{0} e^{ax} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\omega x} dx$$

$$= \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}$$

then take $\hat{f}(x)$ into Poisson theorem.

Theorem 2.3.2 (Sampling theorem)

Let $f(x)$ be continuous and bandlimited on $[-\Omega, \Omega]$. Then, it is uniquely determined by its sampled values at $x = \dfrac{n\pi}{\Omega}$.

※ A function f is called bandlimited on $[-\Omega, \Omega]$ if $\hat{f}(\omega) = 0$ for $|\omega| > \Omega$.

Interpolation formula

$$f(x) = \sum_{n=-\infty}^{\infty} f(nT) \sin c_T (x - nT)$$

where $\sin c_T (x - nT) = \dfrac{\sin(\pi x / T)}{\pi x / T}$

The maximum sampling frequency is $2T$ (Nyquist rate) and $T = \dfrac{\pi}{\Omega}$ is the Nyquist rate.

2.3.3 Fourier Series

1. A periodic function $f(x) = f(x+T)$ can be expanded into a series

$$f(x) = \sum_{n=-\infty}^{\infty} F_n e^{in\frac{2\pi}{T}x},$$

where

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-in\frac{2\pi}{T}x} dx$$

2.3.3 Fourier Series

2. In terms of the trigonometric functions, sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right),$$

where

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx,\ n = 0,1,2,\ldots \quad and$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx,\ n = 0,1,2,\ldots \quad .$$

Example 2.3.5 The Fourier series for $f(x) = \text{sgn}(\cos x)$ as below

$$a_n = \frac{2}{\pi}\int_0^{\pi} \text{sgn}(\cos x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos nx dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos nx dx$$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}, n \in N.$$

$$b_n = 0, n = 1,2,...$$

$$a_0 = 0$$

Therefore,

$$\text{sgn}(\cos x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\sin\frac{\pi}{2}\cos nx = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos(2k+1)x.$$

2.3.4 Discrete Fourier Transform

The discrete Fourier transformation (DFT) of a sequence $\underset{\sim}{f} = \{f_n, n = 0,1,\ldots, N-1\}$ is defined as

$$\underset{\sim}{\mathrm{F}} = \left\{ \sum_{n=0}^{\mathrm{N}-1} f_n w_N^{nk}, k = 0,\ldots, N-1 \right\},$$

where $w_N = e^{-i2\pi/N}$. The inverse is

$$\underset{\sim}{\mathrm{f}} = \left\{ \frac{1}{N} \sum_{k=0}^{\mathrm{N}-1} F_k w_N^{-nk}, n = 0,\ldots, N-1 \right\}.$$

2.4 Heisenberg's uncertainty principle

State:

1. In modeling time-frequency phenomena one cannot be arbitrarily precise in both time and frequency simulataneously.

2. The area of such Heisenberg's box is bounded from below

Hesienberg's box:

If the sides of an imaginary rectangle in the "time-frequency" plane represent the time duration and the spectral bandwidth of a signal.

Symbol

center: $\bar{x} = \frac{1}{\|f\|^2} \int x \, | f(x) |^2 \, dx,$

spectral bandwidth: $(\Delta_f)^2 = \frac{1}{\|f\|^2} \int (x - \bar{x})^2 \, | f(x) |^2 \, dx,$

$*\ \Delta_f$ is called the root mean square (RMS) duration of the signal.

Symbol

(In the frequency domain)

$$\text{center:}\quad \bar{\omega} = \frac{1}{\|\hat{f}\|^2}\int \omega |\hat{f}(\omega)|^2\, d\omega,$$

$$\text{spectral bandwidth:}\quad (\Delta_{\hat{f}})^2 = \frac{1}{\|\hat{f}\|^2}\int \omega^2 |\hat{f}(\omega)|^2\, d\omega.$$

$*\,\Delta_{\hat{f}}$ is called the RMS duration of the bandwidth.

Theorem 2.4.1

Let $f(x) \in L_2(R)$ be such that $xf(x) \in L_2(R)$ is satisfied by both f and $\hat{f}$.

Then,

$$\Delta_f \Delta_{\hat{f}} \geq \frac{1}{2}.$$

the equality is achieved for $f(x) = e^{-at^2}$, $a \geq 0$.

Example 2.4.1

$f(x) = A \cdot 1(x_0 - d/2 \le x \le x_0 + d/2)$, the RMS duration of the bandwidth $\Delta_{\hat{f}}$ is not finite.

$\langle SOL \rangle$:

by Example 2.3.1 $|\hat{\mathrm{f}}(\omega)| = \mathrm{A\ d}\left|\dfrac{\sin\omega\ \mathrm{d}/2}{\omega\ \mathrm{d}/2}\right|^2$ and $\|\hat{f}\|^2 = 2\pi A^2 d$, we obtain

$$(\Delta_{\hat{f}})^2 = \frac{2}{\pi d}\int_R \sin^2\frac{\omega d}{2}\, d\omega = \infty.$$

Thank you!