Statistical Modeling by Wavelets

2 Prerequisites

Hilbert Spaces

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The Inner Product Spaces

A complex vector space H is said to be an inner product space if for any two elements $x,\ y \in \mathrm{H}$ there exists a complex number $\langle x, y \rangle$ (called the inner product of x and y) that satisfies

(1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,for all x, y and $z \in \mathrm{H}$

(3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, for $x,\ y \in \mathrm{H}$ and $\alpha \in C$

(4) $\langle x, x \rangle \geq 0$, for all $x \in \mathrm{H}$

(5) $\langle x, x \rangle = 0$, if and only if $x = 0$

Definition 2.2.1

The sequence $\{x_n\}_{n\in \mathrm{N}}$ is called a Cauchy sequence in H if and only if

$$\|x_m - x_n\| \to 0 \qquad \text{whenever } m, n \to \infty$$

The space H is complete if any Cauchy sequence $\{x_n\}$ is convergent

參考（高微 Fourier Analysis）

Def: A sequence f_n in a inner product space V is a Cauchy sequence

when $\forall \varepsilon > 0, \exists N \; st \; m, n > N \quad \Rightarrow \left| f_n - f_m \right| < \varepsilon$

An inner product space is called complete if every Cauchy sequence

in the space converges

A complete inner product space is called a Hilbert spaces

Example 2.2.2

$L_2(R)$ space (space of all square-integrable functions)

$f \in L_2(R)$ if $\int |f|^2 < \infty$

$\langle f, g \rangle = \int f\, g$, $\|f\| = \sqrt{\int f^2}$

If $f, g \in L_2(C)$, $\langle f, g \rangle = \int f\, \overline{g}$ and $\|f\| = \sqrt{\int f\, \overline{f}}$

Example 2.2.4

A function f belongs to the Lebesgue space $L_p(\mathrm{A}), 1 \le p < \infty$

if $\|f\|_p = \left(\int_{\mathrm{A}} |f(x)|^p\, dx \right)^{1/p} < \infty$ and $\|f\|_\infty = ess \sup\limits_{x \in \mathrm{A}} |f(x)| < \infty$

A linear subspace V is a Hilbert space H is said to be a closed subspace of H if V contains all limiting points

i.e. $x_n \in V$ *and* $\|x_n - x\| \to 0$, *as* $n \to \infty$,then $x \in V$

The orthogonal complement of a subset V of H is defined to be the set $V^{\perp}$ of all elements of H that are orthogonal to every element of V

i.e. $x \in V^{\perp} \Leftrightarrow \langle x, y \rangle = 0, \ \textit{for all} \ y \in V$

Corollary 2.2.1

If V is any subset of the Hilbert space H, then $V^{\perp}$ is a closed subspace of H.

Theorem 2.2.1(Projection theorm)

If V is a closed subspace of the Hilbert space H and $x \in H$, then

(1)There is a unique element $\hat{x} \in V$ such that $\left\| x - \hat{x} \right\| = \inf_{y \in V} \|x - y\|$

(2) $\hat{x} \in V$ and $\left\| x - \hat{x} \right\| = \inf_{y \in V} \|x - y\|$ if and only if $\hat{x} \in V$,and $x - \hat{x} \in V^{\perp}$

The closed span $\overline{span}\{x_\lambda, \lambda \in \Lambda\}$ of any subset $\{x_\lambda, \lambda \in \Lambda\}$ of H is defined to be smallest closed subspace of H that contains each element $x_\lambda, \lambda \in \Lambda$

Definition 2.2.2

A set $\{e_\lambda, \lambda \in \Lambda\}$ of elements from H is orthonormal

if $\langle e_s, e_t \rangle = \delta_{s,t}, \ s,t \in \Lambda$

Let $\{e_1, e_2, \cdots e_n\}$ be an orthonormal subset of H and let $\mathrm{M} = \overline{span}\{e_1, e_2, \cdots, e_n\}$

Then

(1) For any $x \in H$, $\mathrm{Pr}\, oj_M X = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$

(2) For any $(a_1, a_2 \cdots a_n)$ and any $x \in H$

$$\left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\| \le \left\| x - \sum_{i=1}^{n} a_i e_i \right\|$$

with equality only for $a_i = \langle x, e_i \rangle$

(3) $\sum_{i=1}^{n} \left| \langle x, e_i \rangle \right|^2 \le \|x\|^2$ (Bessel's inequality)

Definition

Hilbert space H is separable if $\overline{span}\{e_\lambda, \lambda \in \Lambda\}$ and the set $\{e_\lambda, \lambda \in \Lambda\}$ is finite or countable. Such a set is called a basis.

Theorem 2.2.2

Let H be a separable Hilbert space with a basis $\{e_n, n \in \mathrm{N}\}$. Then

(1) For any $x \in H$ and $\varepsilon > 0$, one can find N large enough and constants

$a_1, a_2, \cdots, a_N$ such that $\left\| x - \sum_{n=1}^{N} a_n e_n \right\| < \varepsilon$

(2) $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$

(3) (Parseval's identity) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$

(4) For any $x, y \in H$, $\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \langle e_n, y \rangle$

(5) If x=0 then for all n: $\langle x, e_n \rangle = 0$

Definition:

A countable set $\{f_n \mid n \in N\}$ of elements f_n from separable Hilbert space H, constitutes a **frame** iff

there are two constants A and B, such that for every $x \in H$

$$A\|x\|^2 \le \sum_{n\in Z} \left|\langle x, f_n \rangle\right|^2 \le B\|x\|^2 \quad (0 < A \le B < \infty)$$

Constants A and B are called frame bounds.

Note: (1) If A=B$\geq$1, the frame is called **tight**.

(2)The frame is called **exact** if it is minimal

i.e. if it ceases to be a frame whenever any single element is removed from the set.

(3)When $\{f_n \mid n \in N\}$ constitutes a tight frame, $x \in H$ can be uniquely reconstructed from $\{a_1, a_2, \cdots a_n \cdots\}$ from $x = \sum_n a_n f_n$

Example 2.2.8

If $\{e_1, e_2, \cdots e_n \cdots\}$ is an orthonormal basis for a space H, then

(a) $\{e_1, e_1, e_2, e_2 \cdots, e_n, e_n \cdots\}$ is an inexact, tight frame with frame bounds

A=B=2

(b) $\left\{e_1, \frac{e_2}{2}, \cdots \frac{e_n}{n} \cdots\right\}$ is a complete, orthogonal sequence, but not a frame

(c) $\{2e_1, e_2, e_3, \cdots e_n, \cdots\}$ is an exact, nontight frame with bounds A=1 and B=4

(d) $\left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}} \cdots\right\}$ is an inexact, tight frame with bounds

A=B=1,but no nonredundants subsequence is a frame.

Definition

A countable set $\{f_n \mid n \in N\}$ of elements f_n from a separable Hilbert space H, constitutes a **Riesz basis** if for any $x \in H$ there is a unique representation

$x = \sum_n a_n f_n$ and $A \cdot \sum_n |a_n|^2 \leq \left\| \sum_n a_n f_n \right\|^2 \leq B \cdot \sum_n |a_n|^2$

for some constants A and B, $(0 < A \leq B < \infty)$

NOTE: (1) The Riesz basis reduces to an orthonomal basis when A=B=1.

(2) Every Riesz basis is a frame, but the contray is false.

(3) Every exact frame is a Riesz basis.

Definition 2.2.3

A series $\sum_{n \in S} a_n$ converges unconditionally if for every “1-1”and”onto” map $\pi : N \to S$ the series $\sum_{k=1}^{\infty} a_{\pi(k)}$ converges.

Definition 2.2.4

A basis $\beta = \{e_i, i \in N\}$ is called unconditional for a space V

if and only if

the sum in unique representation $x = \sum_{i=1}^{\infty} b_i e_i$ of an element $x \in V$

converges unconditionally.

2.2.3 Reproducing Kernel Hilbert Spaces

A function of two variables x and y, K(x, y) is called a reproducing kernel function for the function space H if

(1) For a fixed y, K(x, y) is a function in H

(2) For every function $f \in H$ and every y, K has the reproducing property,

$$f(y) = \langle f(x), K(x, y) \rangle$$

Theorem 2.2.3

Let V be a subspace of L_2 and let $\{e_1, e_2, e_3, \cdots e_n, \cdots\}$ be a complete orthonormal basis of V. Then V is a reproducing kernel Hilbert space with a kernel(sometimes called the Bergman kernel)

$$K(x, y) = \sum_n e_n(x) e_n(y) \qquad (2.2)$$

For any function $f \in V$,

$$f(y) = \int f(x) K(x, y)\, dx$$