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第十章：兩個母體的比較

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Summary

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Introduction

In this chapter, we extend the discussion of statistical inference by developing interval estimates and conducting hypothesis tests for situations involving the difference between two population means or two population proportions.

For example, we may want to

- Develop an interval estimate of the difference in mean starting salary between a population of males and a population of females.
- Conduct a hypothesis test to determine whether there is a difference in the proportion of defective parts between the populations of parts produced by two suppliers.

We begin our discussion of inference about the difference between two population means by first considering the case of population standard deviations assumed known, and then repeating the process for the case of population standard deviations assumed unknown.

Finally, we consider the inference of the difference between two population means when using a matched sample design and the difference between two population proportions.

10.1 Interval Estimation of $\mu_1 - \mu_2$: σ_1 and σ_2 Known

The interval estimate of the difference between two population means $(\mu_1 - \mu_2)$, when σ_1 and σ_2 are known, can be written as follows.

$$\boldsymbol{\bar{x}_1 - \bar{x}_2 \pm z_{\alpha/2}\sigma_{\bar{x}_1-\bar{x}_2}}$$

Where

$\bar{x}_1 - \bar{x}_2$ is the point estimator of $\mu_1 - \mu_2$

$1 - \alpha$ is the confidence coefficient

$z_{\alpha/2}$ is the critical value such that $P\left(z \geq z_{\alpha/2}\right) = 1 - \alpha/2$

$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ is the standard error of $\bar{x}_1 - \bar{x}_2$ ($*$see notes)

$z_{\alpha/2}\sigma_{\bar{x}_1-\bar{x}_2}$ is the margin of error

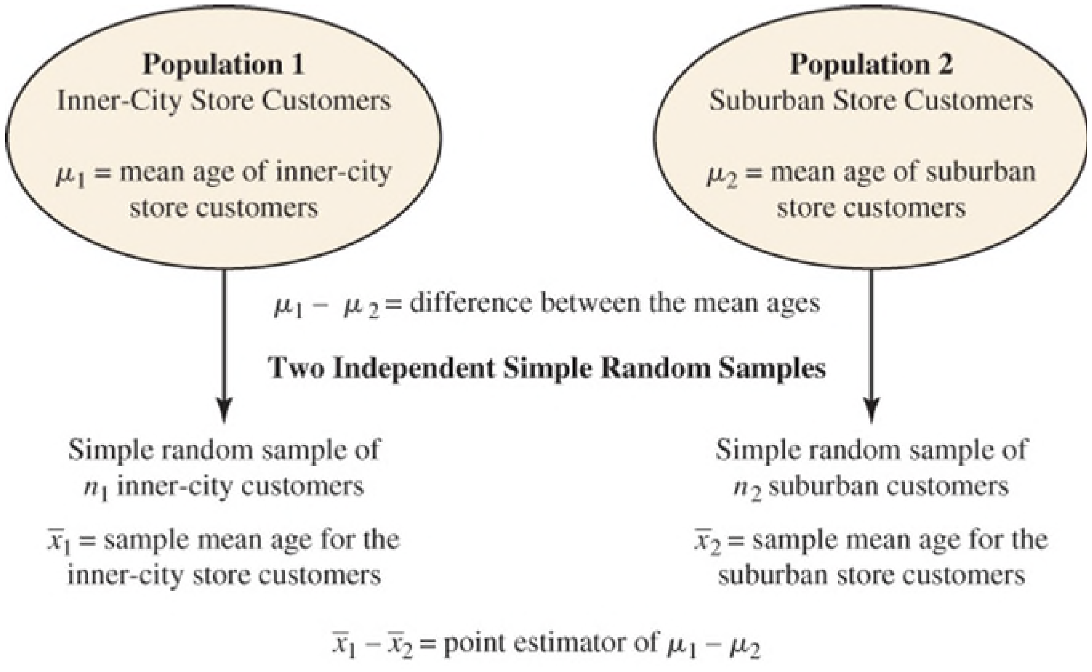
10.1 The Graystone Department Stores Example

The regional manager at Graystone Department Stores wants to investigate the difference between the mean ages of the customers who shop at two of their stores, one in the inner-city (index 1) and one in a suburban area (index 2.)

Population standard deviations from previous studies are known as $\sigma_1 = 9$ years and $\sigma_2 = 10$ years.

Two independent random samples of Graystone customers provided the following information.

	Inner City	**Suburban**
Sample size	$n_1 = 36$	$n_2 = 49$
Sample mean	$\bar{x}_1 = 40$ years	$\bar{x}_2 = 35$ years



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10.1 Interval Estimate of the Difference Between the Mean Ages at Graystone Dept. Stores

The point estimate of the difference between the two population mean ages is

$$\bar{x}_1 - \bar{x}_2 = 40 - 35 = 5 \text{ years}$$

The standard error of $\bar{x}_1 - \bar{x}_2$ is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{9^2}{36} + \frac{10^2}{49}} = 2.07 \text{ years}$$

Using 95% confidence and $z_{\alpha/2} = z_{.025} = 1.96$, we have

$$\bar{x}_1 - \bar{x}_2 \pm z_{\alpha/2}\sigma_{\bar{x}_1 - \bar{x}_2} = 5 \pm 1.96(2.07) = 5 \pm 4.06 \text{ years}$$

Thus, the margin of error is 4.06 years, and the 95% confidence interval estimate of the difference between the two population means is $5 - 4.06 = 0.94$ years to $5 + 4.06 = 9.06$ years.

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10.1 Hypothesis Tests About $\mu_1 - \mu_2$

In general, a hypothesis test about the difference between two population means ($\mu_1 - \mu_2$) must take one of the following three forms.

One-tailed (Lower-tail)	**One-tailed (Upper-tail)**	**Two-tailed**
$H_0: \mu_1 - \mu_2 \geq D_0$	$H_0: \mu_1 - \mu_2 \leq D_0$	$H_0: \mu_1 - \mu_2 = D_0$
$H_a: \mu_1 - \mu_2 < D_0$	$H_a: \mu_1 - \mu_2 > D_0$	$H_a: \mu_1 - \mu_2 \neq D_0$

Where D_0 denotes the hypothesized difference between μ_1 and μ_2.

Using the two-tailed test as an example, when $D_0 = 0$, as it is in most cases, the null hypothesis is $H_0: \mu_1 - \mu_2 = 0$. In this case, the null hypothesis is that μ_1 and μ_2 are equal.

The rejection of H_0 leads to the conclusion that $H_a: \mu_1 - \mu_2 \neq 0$ is true; that is, μ_1 and μ_2 are not equal.

10.1 Test Statistic for Hypothesis Tests About $\mu_1 - \mu_2$: σ_1 and σ_2 Known

When the sample sizes are large enough, the distribution of $\bar{x}_1 - \bar{x}_2$ can be described by a normal distribution (*see notes.)

In this case, the test statistic for the difference between two population means ($\mu_1 - \mu_2$), when σ_1 and σ_2 are known, can be written as

$$\boldsymbol{z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}}$$

Where

$\bar{x}_1 - \bar{x}_2$ is the point estimator of $\mu_1 - \mu_2$

D_0 is the hypothesized difference between μ_1 and μ_2

$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ is the standard error of $\bar{x}_1 - \bar{x}_2$

10.1 A Quality Assessment of Two Training Centers

As part of a study to evaluate differences in education quality, a standardized examination is given to individuals who are trained at two training centers.

The difference between the mean examination scores is used to assess quality differences between training center A (index 1) and training center B (index 2.)

Historical standardized examinations always resulted in an examination score standard deviation near 10 points. Thus, we will assume $\sigma_1 = 10$ and $\sigma_2 = 10$.

DATAfile: *ExamScores*

Independent random samples of individuals from training centers A and B provided the following information.

	Center A	**Center B**
Sample size	$n_1 = 30$	$n_2 = 40$
Sample mean	$\bar{x}_1 = 82$	$\bar{x}_2 = 78$

An $\alpha = 0.05$ level of significance is specified for the study.

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10.1 A Two-Tailed Test About $\mu_1 - \mu_2$: σ_1 and σ_2 Known

We begin with the tentative assumption that no difference exists between the training quality provided at the two centers. Hence, the null hypothesis is that $\mu_1 - \mu_2 = 0$, and its rejection leads to the conclusion that a quality differential exists between center A and center B.

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 \neq 0$$

Calculations of the test statistic follow.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sigma_{\bar{x}_1 - \bar{x}_2}} = \frac{(82 - 78) - 0}{\sqrt{10^2/30 + 10^2/40}} = 1.66$$

The *p*-value for a two-tailed test is: $p\text{−value} = 2P(z \geq 1.66) = 2[1 - P(z < 1.66)] = 0.0970$

Thus, because $p\text{−value} > \alpha$, we cannot reject H_0. The sample results do not provide sufficient evidence to conclude that the training centers differ in quality.

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10.2 Interval Estimation of $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown

The interval estimate of the difference between two population means $(\mu_1 - \mu_2)$, when σ_1 and σ_2 are unknown, can be written as follows.

$$\boldsymbol{\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2}\sigma_{\bar{x}_1-\bar{x}_2}}$$

Where

$\bar{x}_1 - \bar{x}_2$ is the point estimator of $\mu_1 - \mu_2$

$1 - \alpha$ is the confidence coefficient

$$df = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1 - 1) + (s_2^2/n_2)^2/(n_2 - 1)}$$ (rounded down to an integer)

$t_{\alpha/2}$ is the critical value such that $P(t \geq t_{\alpha/2}) = 1 - \alpha/2$, with df degrees of freedom

$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ is the standard error of $\bar{x}_1 - \bar{x}_2$

$t_{\alpha/2}\sigma_{\bar{x}_1-\bar{x}_2}$ is the margin of error

10.2 A Checking Account Balance Study at Clearwater National Bank

Clearwater National Bank would like to estimate the difference between the mean checking account balance maintained by the population of customers at two of its branches: Cherry Grove (index 1) and Beechmont (index 2.)

DATAfile: *CheckAcct*

Information on the checking account balance for two independent random samples of checking accounts from the Cherry Grove and Beechmont branches was recorded (*see notes.)

Samples statistics were calculated as follows.

	Cherry Grove	**Beechmont**
Sample size	$n_1 = 28$	$n_2 = 22$
Sample mean	$\bar{x}_1 = \$1{,}025$	$\bar{x}_2 = \$910$
Sample standard deviation	$s_1 = \$150$	$s_2 = \$125$

Develop an interval estimate of the difference in mean checking account balance between the two branches at a 95% confidence level.

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10.2 Interval Estimate of the Mean Checking Account Balance Difference at Clearwater National Bank

The standard error of $\bar{x}_1 - \bar{x}_2$ is

$$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{150^2}{28} + \frac{125^2}{22}} = 38.9$$

The degrees of freedom (*see notes) for $t_{\alpha/2}$ are

$$df = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1-1) + (s_2^2/n_2)^2/(n_2-1)} = \frac{(150^2/28 + 125^2/22)^2}{(150^2/28)^2/27 + (125^2/22)^2/21} = 47.8$$

Using 95% confidence and $df = 47$, it follows that $t_{\alpha/2} = t_{.025} = 2.012$, and we have

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2}\sigma_{\bar{x}_1-\bar{x}_2} = (1025 - 910) \pm 2.012(38.9) = 115 \pm 78$$

Thus, the margin of error is \$78, and the 95% confidence interval estimate of the difference between the two population means is $115 - 78 = \$37$ to $115 + 78 = \$193$.

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10.2 Test Statistic for Hypothesis Tests About $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown

When σ_1 and σ_2 are unknown, we use s_1 as an estimator of σ_1, and s_2 as an estimator of σ_2.

The test statistic for the difference between two population means $(\mu_1 - \mu_2)$, when σ_1 and σ_2 are unknown, can be described by a *t* distribution with *df* degrees of freedom.

$$\boldsymbol{t} = \frac{(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2) - \boldsymbol{D}_0}{\boldsymbol{\sigma}_{\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2}}$$

Where

$\bar{x}_1 - \bar{x}_2$ is the point estimator of $\mu_1 - \mu_2$, and D_0 the hypothesized difference between μ_1 and μ_2

$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\sigma_1^2/n_1 + \sigma_1^2/n_1}$ is the standard error of $\bar{x}_1 - \bar{x}_2$

$$df = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1 - 1) + (s_2^2/n_2)^2/(n_2 - 1)}$$ (rounded down to an integer)

10.2 A New Software Project Completion Time Study

A new software package was developed to help system analysts reduce the completion time of a project consisting of the design, development, and implementation of an information system.

DATAfile: *SoftwareTest*

To assess the new software, information on the project completion time using the current technology (index 1) and the new software (index 2) was recorded for two independent random samples of 12 system analysts each.

Samples statistics were calculated as follows.

	Current Technology	**New Software**
Sample size	$n_1 = 12$	$n_2 = 12$
Sample mean	$\bar{x}_1 = 325$ hours	$\bar{x}_2 = 286$ hours
Sample standard deviation	$s_1 = 40$ hours	$s_2 = 44$ hours

Conduct a hypothesis test to assess whether the new software decreases mean project completion time at a significance level, $\alpha = 0.05$.

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10.2 An Upper Tail Test About $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown

Because we are testing for a decrease in mean project completion time for the new software (index 2) with respect to the current technology (index 1), we write $H_a: \mu_1 - \mu_2 > 0$.

We set the hypotheses for an upper tail test with the hypothesized mean difference, $D_0 = 0$, as

$$H_0: \mu_1 - \mu_2 \leq 0$$

$$H_a: \mu_1 - \mu_2 > 0$$

Calculations of the test statistic follow.

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sigma_{\bar{x}_1 - \bar{x}_2}} = \frac{(325 - 286) - 0}{\sqrt{40^2/12 + 44^2/12}} = 2.27$$

The test statistic is described by a *t* distribution with 21 degrees of freedom.

$$df = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1 - 1) + (s_2^2/n_2)^2/(n_2 - 1)} = \frac{(40^2/12 + 44^2/12)^2}{(40^2/12)^2/11 + (44^2/12)^2/11} = 21.8$$

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10.2 *p*-Value Approach to a One-Tailed Test About $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown

The *p*-value is the upper tail of a *t* distribution with 21 degrees of freedom.

$$p\text{−value} = P(t \geq 2.27)$$

Using Table 2 in Appendix B, the *t* distribution with 21 degrees of freedom provides the following information.

Area in upper tail	**0.20**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
***t* Value ($df = 21$)**	0.859	1.323	1.721	2.080	2.518	2.831

Because $t = 2.27$ is between 2.080 and 2.518, the values in the "Area in Upper Tail" row show that the *p*-value range must be: $0.01 \leq p\text{−value} \leq 0.025$.

We can also use Excel to compute the exact *p*-value as: $=\text{T.DIST.RT}(2.27, 21) = 0.0169$

Because p−value $\leq \alpha$, we can reject the null hypothesis and conclude that the new software decreases mean project completion time.

10.3 Matched Sample Design

A manufacturing company wants to determine whether two different methods to perform a production task have the same population mean completion time.

We set the null hypothesis based on the assumption that method 1 and method 2 have the same population mean completion time and challenge it in the alternative hypothesis.

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 \neq 0$$

We have two choices of sampling procedure:

- *Independent sample design*: two simple random samples of workers are selected. One sample of workers is assigned to use method 1, and the other one to use method 2.
- ***Matched sample*** *design*: one simple random sample of workers using both methods, one at a time, is selected. The order of the methods is assigned randomly to the workers.

The matched sample design often leads to a smaller sampling error because by testing the two methods using the same workers, the variation in worker's efficiency is eliminated (*see notes.)

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10.3 Hypothesis Tests Involving Matched Samples

DATAfile: *Matched*

A simple random sample of six workers is selected, and the pair of data values (x_1 and x_2) for the production task completion time in minutes using methods 1 and 2 are recorded.

We define a variable difference *d* as

$$d = x_1 - x_2$$

The key to the analysis of the matched sample design is that we only consider the values of the difference *d* for the six workers:

$$0.6, -0.2, 0.5, 0.3, 0.0, \text{and } 0.6$$

With μ_d as the population mean difference, we can write the hypotheses to a two-tailed test as

$$H_0: \mu_d = D_0$$
$$H_a: \mu_d \neq D_0$$

In this example, because we are looking for a difference between the two methods, $D_0 = 0$.

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10.3 Test Statistic for Hypothesis Tests Involving Matched Samples

The test statistic for μ_d, can be described by a *t* distribution with $n - 1$ degrees of freedom.

$$\boldsymbol{t} = \frac{\boldsymbol{\bar{d} - D_0}}{\boldsymbol{\sigma_{\bar{d}}}}$$

Where

$\bar{d} = \frac{\sum d_i}{n}$ is the point estimator of μ_d

D_0 is the hypothesized mean difference

$\sigma_{\bar{d}} = s_d/\sqrt{n}$ is the standard error of $\bar{d}$

$s_d = \sqrt{\sum (d_i - \bar{d})^2/(n-1)}$ is the sample standard deviation of the difference

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10.3 *p*-Value Approach to a Test About Matched Samples

Calculations lead to $n = 6$, $\bar{d} = 0.30$, and $s_d = 0.335$. Thus, the test statistic is

$$t = \frac{\bar{d} - D_0}{\sigma_{\bar{d}}} = \frac{\bar{d} - D_0}{s_d/\sqrt{n}} = \frac{0.30 - 0}{0.335/\sqrt{6}} = 2.20$$

The *p*-value for this two-tailed test is twice the upper tail of a *t* distribution with 5 degrees of freedom (*see notes.) Using Table 2 in Appendix B, we have

Area in upper tail	**0.20**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
***t* Value ($df = 5$)**	0.920	1.476	2.015	2.571	3.365	4.032

For a two-tailed test, the *p*-value is twice the 0.025 to 0.05 range identified in the "Area in Upper Tail" row for $t = 2.20$. Thus, we have: $0.05 \leq p\text{–value} \leq 0.10$.

We can also use Excel to compute the exact *p*-value as: $= \text{T.DIST.2T}(2.20, 5) = 0.0791$

Because p–value $> \alpha$, we do not reject the null hypothesis and cannot conclude that the two production methods have a different mean completion time.

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10.3 Interval Estimation Involving Matched Samples

We can obtain an interval estimate of the population mean completion time difference between the two methods using the single population methodology from Chapter 8.

$$\bar{\boldsymbol{d}} \pm \boldsymbol{t}_{\alpha/2}\boldsymbol{s}_{d}/\sqrt{\boldsymbol{n}}$$

With 95% confidence and $n - 1 = 5$ degrees of freedom, Table 2 in Appendix B can be used to obtain $t_{.025} = 2.571$.

Area in upper tail	**0.20**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
***t* Value ($df = 5$)**	0.920	1.476	2.015	2.571	3.365	4.032

Alternatively, the Excel equation =T.INV(1−0.025,5) will also generate the value 2.571.

We compute an interval estimate of the population mean completion time difference as

$$\bar{d} \pm t_{.025}s_d/\sqrt{n} = 0.3 \pm 2.571\left(0.335/\sqrt{6}\right) = 0.3 \pm 0.35$$

We are 95% confident that the population mean completion time difference between the two production methods is between $0.3 - 0.35 = -0.05$ minutes and $0.3 + 0.35 = 0.65$ minutes.

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10.4 Interval Estimation of $p_1 - p_2$

The interval estimate of the difference between two population proportions $(p_1 - p_2)$, is based on the sampling distribution of $\bar{p}_1 - \bar{p}_2$, which can be approximated by a normal distribution so long as $n_1 p_1$, $n_1(1 - p_1)$, $n_2 p_2$, and $n_2(1 - p_2)$ are all greater than or equal to 5.

In such case, the confidence interval of $p_1 - p_2$ can be written as follows.

$$\bar{\boldsymbol{p}}_1 - \bar{\boldsymbol{p}}_2 \pm \boldsymbol{z}_{\alpha/2} \boldsymbol{\sigma}_{\bar{\boldsymbol{p}}_1 - \bar{\boldsymbol{p}}_2}$$

Where

$\bar{p}_1 - \bar{p}_2$ is the point estimator of $p_1 - p_2$

$z_{\alpha/2}$ is the critical value such that $P(z \geq z_{\alpha/2}) = 1 - \alpha/2$

$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{n_1} + \frac{\bar{p}_2(1 - \bar{p}_2)}{n_2}}$ is the standard error of $\bar{p}_1 - \bar{p}_2$

$z_{\alpha/2} \sigma_{\bar{p}_1 - \bar{p}_2}$ is the margin of error

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10.4 A Tax Preparation Example

A tax preparation firm is interested in comparing the quality of work at two of its regional offices.

A random sample of tax returns prepared at each of the two offices is selected and every tax return in both samples is checked for accuracy.

The analysis of the sampled tax returns at the two regional offices provided the following information.

	Office 1	**Office 2**
Sample size	$n_1 = 250$	$n_2 = 300$
Number of returns with errors	$x_1 = 35$	$x_2 = 27$
Sample proportion	$\bar{p}_1 = 35/250 = 0.14$	$\bar{p}_2 = 27/300 = 0.09$

The point estimate of the difference between the proportions of erroneous tax returns for the two populations is $\bar{p}_1 - \bar{p}_2 = 0.14 - 0.09 = 0.05$.

Thus, we estimate that office 1 has a 0.05, or 5%, greater error rate than office 2 does.

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10.4 Estimation of the Difference Between the Proportions of Tax Return Errors

The standard error of $\bar{p}_1 - \bar{p}_2$ is

$$\sigma_{\bar{p}_1-\bar{p}_2} = \sqrt{\frac{\bar{p}_1(1-\bar{p}_1)}{n_1} + \frac{\bar{p}_2(1-\bar{p}_2)}{n_2}} = \sqrt{\frac{0.14(1-0.14)}{250} + \frac{0.09(1-0.09)}{300}} = 0.0275$$

Using 90% confidence and $z_{\alpha/2} = z_{.05} = 1.645$, we have

$$\bar{p}_1 - \bar{p}_2 \pm z_{\alpha/2}\sigma_{\bar{p}_1-\bar{p}_2} = 0.05 \pm 1.645(0.0275) = 0.05 \pm 0.045$$

Thus, the margin of error is 0.045, and the 90% confidence interval estimate of the difference between the two population proportions is $0.05 - 0.045 = 0.005$ to $0.05 + 0.045 = 0.095$.

We are 90% confident that the difference in error rates between office 1 and office 2 is between 0.5% and 9.5%.

10.4 Hypothesis Tests About $p_1 - p_2$

In general, a hypothesis test about the difference between two population proportions $(p_1 - p_2)$ must take one of the following three forms.

One-tailed (Lower-tail)	One-tailed (Upper-tail)	Two-tailed
$H_0: p_1 - p_2 \geq D_0$	$H_0: p_1 - p_2 \leq D_0$	$H_0: p_1 - p_2 = D_0$
$H_a: p_1 - p_2 < D_0$	$H_a: p_1 - p_2 > D_0$	$H_a: p_1 - p_2 \neq D_0$

Where D_0 denotes the hypothesized difference between p_1 and p_2.

Using an upper-tail test as an example, when $D_0 = 0$, as it is in most cases, we assume the null hypothesis to be true and that $p_1 - p_2 \leq 0$; that is, p_1 is less than or equal to p_2.

The rejection of H_0 leads to the conclusion that $H_a: p_1 - p_2 > 0$ is true; that is, p_1 is greater than p_2.

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10.4 Test Statistic for Hypothesis Tests About $p_1 - p_2$

The test statistic of the difference between p_1 and p_2, is based on the sampling distribution of $\bar{p}_1 - \bar{p}_2$, which can be approximated by a normal distribution so long as $n_1 p_1$, $n_1(1 - p_1)$, $n_2 p_2$, and $n_2(1 - p_2)$ are all greater than or equal to 5.

In such case, the test statistic for $p_1 - p_2$ can be written as

$$\boldsymbol{z = \frac{(\bar{p}_1 - \bar{p}_2) - D_0}{\sigma_{\bar{p}_1 - \bar{p}_2}}}$$

Where

D_0 is the hypothesized difference between p_1 and p_2

$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\bar{p}(1 - \bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$ is the standard error of $\bar{p}_1 - \bar{p}_2$ ($*$see notes.)

$\bar{p} = \frac{n_1 \bar{p}_1 + n_2 \bar{p}_2}{n_1 + n_2}$ is the pooled estimator of p, a weighted average of $\bar{p}_1$ and $\bar{p}_2$

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10.4 A Two-Tailed Test About $p_1 - p_2$

We assume that $H_0\colon p_1 - p_2 = 0$ and challenge the assumption in the alternative hypothesis.

$$H_0\colon p_1 - p_2 = 0$$

$$H_a\colon p_1 - p_2 \neq 0$$

The pooled estimator of *p* is:

$$\bar{p} = (n_1\bar{p}_1 + n_2\bar{p}_2)/(n_1 + n_2) = [250(0.14) + 300(0.09)]/(250 + 300) = 0.1127$$

Calculations of the test statistic follow.

$$z = \frac{(\bar{p}_1 - \bar{p}_2) - D_0}{\sigma_{\bar{p}_1 - \bar{p}_2}} = \frac{(0.14 - 0.09) - 0}{\sqrt{0.1127(1 - 0.1127)(1/250 + 1/300)}} = 1.85$$

The *p*-value for a two-tailed test is: $p\text{–value} = 2P(z \geq 1.85) = 2[1 - P(z < 1.85)] = 0.0322$

Thus, because $p\text{–value} \leq \alpha$, we reject H_0. The sample results provide sufficient evidence to conclude that a difference exists between the proportion of tax return errors at the two centers.

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Summary

- In this chapter, we discussed procedures for developing interval estimates and conducting hypothesis tests involving two populations.
- First, we showed how to make inferences about the difference between two population means when independent simple random samples are selected.
  - When the population standard deviations, σ_1 and σ_2, can be assumed known, the standard normal distribution z is used to make inferences.
  - When the population standard deviations, σ_1 and σ_2, are unknown and estimated by the sample standard deviations, s_1 and s_2, the t distribution is used instead.
- We then discussed inferences about the difference between two population means for the matched sample design, in which the difference between the paired data values is used.
- Finally, we presented how to make inferences about the difference between two population proportions using the standard normal distribution z.

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