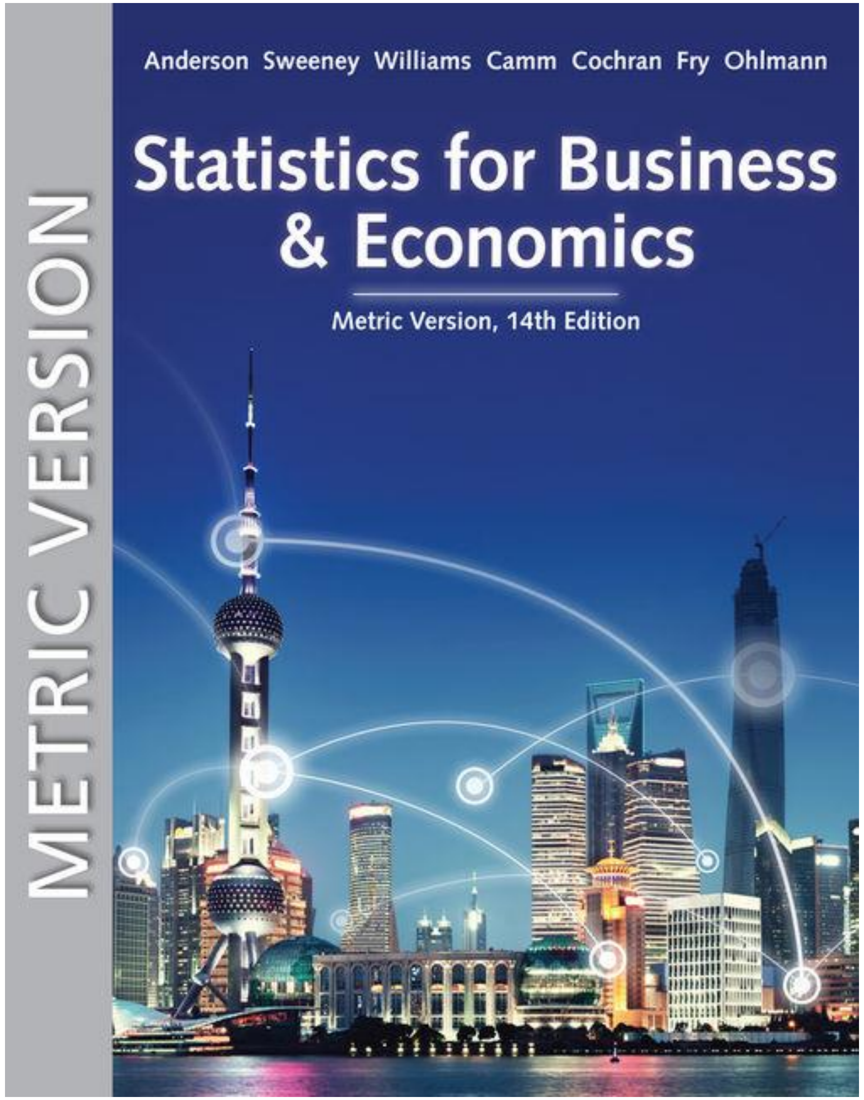
Statistics for Business and Economics (14e) Metric Version

Chapter 13 （變異數分析）



Chapter 13 - Experimental Design and Analysis of Variance

An Introduction to Experimental Design and Analysis of Variance (1 of 3)

- Statistical studies can be classified as being either experimental or observational.
- In an experimental study, one or more factors are controlled so that data can be obtained about how the factors influence the variables of interest.
- In an observational study, no attempt is made to control the factors.
- Cause-and-effect relationships are easier to establish in experimental studies than in observational studies.
- Analysis of variance (ANOVA) can be used to analyze the data obtained from experimental or observational studies.

An Introduction to Experimental Design and Analysis of Variance (2 of 3)

In this chapter, three types of experimental designs are introduced:

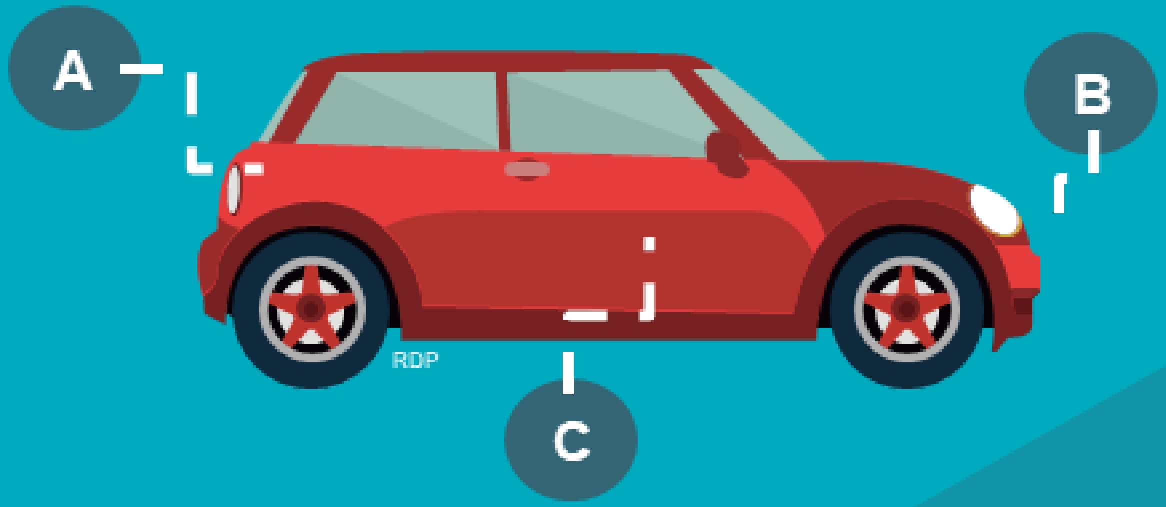
- A completely randomized design（完全隨機設計）
- A randomized block design（隨機區塊設計）
- A factorial (design) experiment（因子設計實驗）

An Introduction to Experimental Design and Analysis of Variance (3 of 3)

- A factor （因子）is a variable that the experimenter has selected for investigation.
- A treatment （處理）is a level of a factor.
- Experimental units （實驗單位）are the objects of interest in the experiment.
- A completely randomized design is an experimental design in which the treatments are randomly assigned to the experimental units.

Analysis of Variance (ANOVA)

<u>**Bundling Campaign Contributions**</u>



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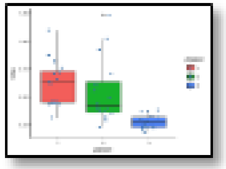
Analysis of Variance



A method to determine whether there are any statistically significant differences between the population means of three or more independent (unrelated) groups

Definition

Characteristic

- *respondent / dependent variable is a continuous variable*
- *independent variable is categorical variable with two or more group*
- *the data of all groups are normally distributed and homogeneous*
- *two or more groups are not related*

Analysis of Variance: A Conceptual Overview (1 of 4)

- Analysis of Variance (ANOVA) can be used to test for the equality of three or more population means.
- Data obtained from observational or experimental studies can be used for the analysis.
- We want to use the sample results to test the following hypotheses:

$$H_0\colon \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$$

H_a: Not all population means are equal.

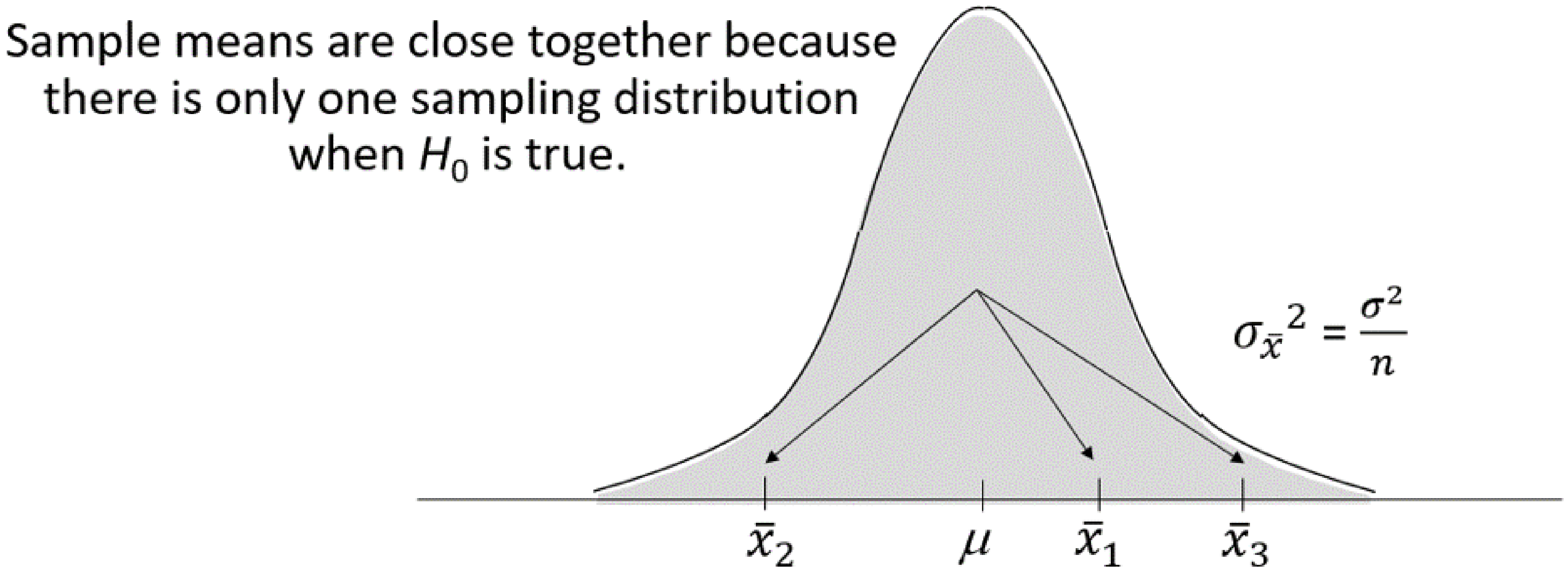
Analysis of Variance: A Conceptual Overview (2 of 4)

- If H_0 is rejected, we cannot conclude that all population means are different.
- Rejecting H_0 means that at least two population means have different values.

- Assumptions for Analysis of Variance
  1. For each population, the response (dependent) variable is normally distributed.
  2. The variance of the response variable, denoted σ^2, is the same for all of the populations.
  3. The observations must be independent.

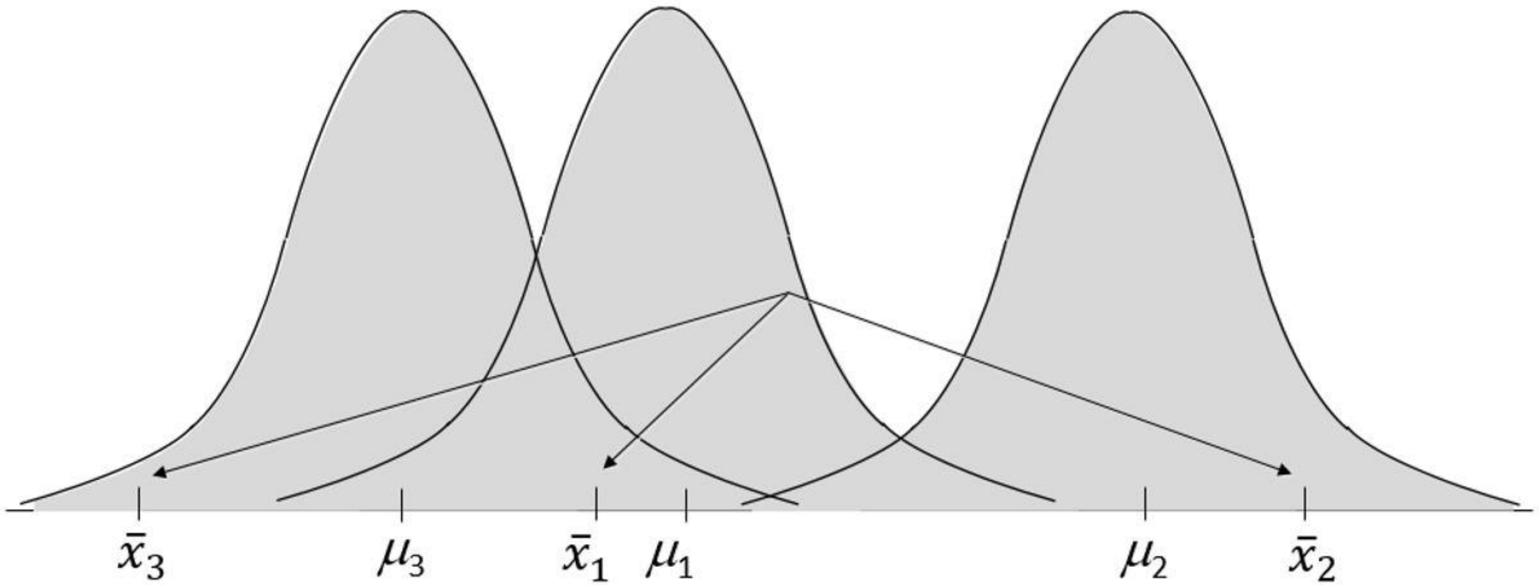
Analysis of Variance: A Conceptual Overview (3 of 4)

Sampling distribution of $\bar{x}$, given H_0 is true.



Analysis of Variance: A Conceptual Overview (4 of 4)

Sampling distribution of $\bar{x}$, given H_0 is false.



Sample means come from different sampling distributions and are not as close together when H_0 is false.

Analysis of Variance and the Completely Randomized Design

- Between-Treatments Estimate of Population Variance
- Within-Treatments Estimate of Population Variance
- Comparing the Variance Estimates: The *F* Test
- ANOVA Table

Between-Treatments Estimate of Population Variance σ^2

The estimate of σ^2 based on the variation of the sample means is called the mean square due to treatments and is denoted by MSTR.

$$\text{MSTR} = \frac{\sum_{j=1}^{k} n_j \left(\bar{x}_j - \bar{\bar{x}}\right)^2}{k-1}$$

Numerator is called the sum of squares due to treatments (SSTR).

Denominator is the degrees of freedom associated with SSTR.

Within-Treatments Estimate of Population Variance σ²

The estimate of σ^2 based on the variation of the sample observations within each sample is called the mean square error and is denoted by MSE.

$$\text{MSE} = \frac{\sum_{j=1}^{k}(n_j - 1)s_j^{\,2}}{n_T - k}$$

Numerator is called the sum of squares due to error (SSE).

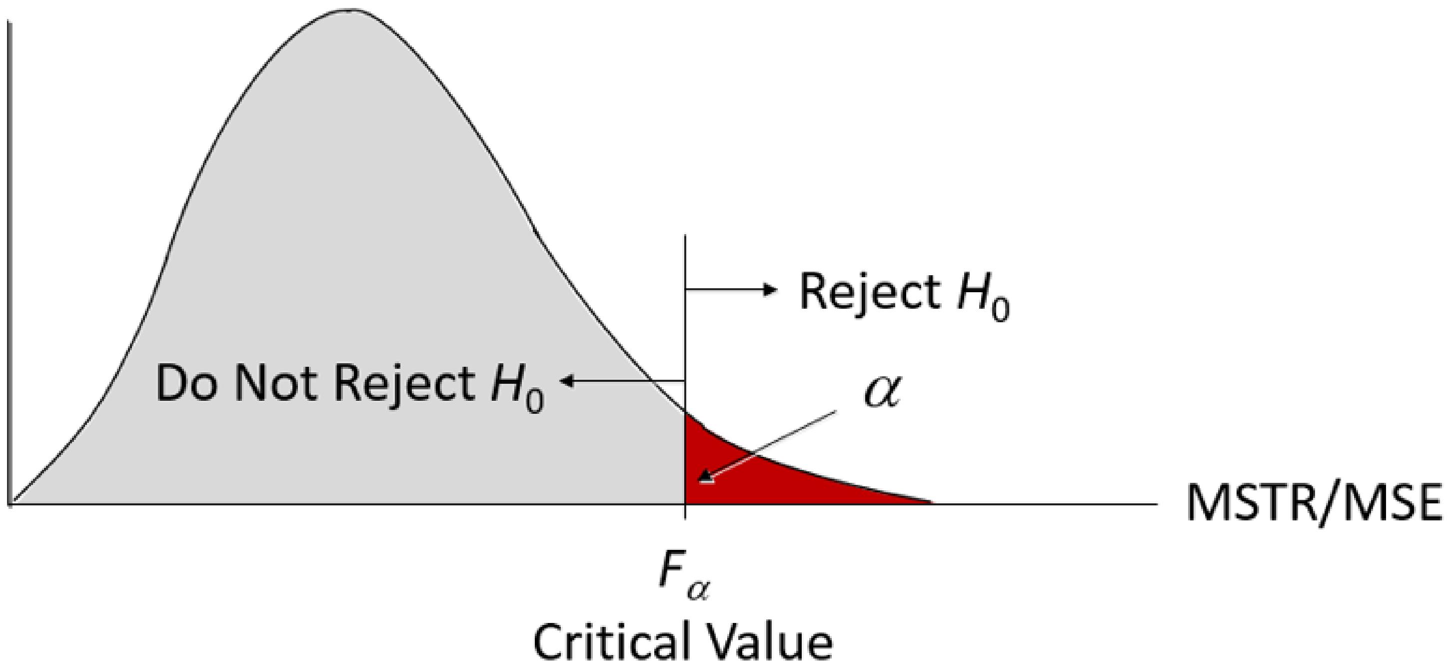
Denominator is the degrees of freedom associated with SSE.

Comparing the Variance Estimates: The *F* Test (1 of 2)

- If the null hypothesis is true and the ANOVA assumptions are valid, the sampling distribution of MSTR/MSE is an F distribution with MSTR degrees of freedom equal to k – 1 and MSE degrees of freedom equal to $n_T - k$.
- If the means of the k populations are not equal, the value of MSTR/MSE will be inflated because MSTR overestimates σ^2.
- Hence, we will reject H_0 if the resulting value of MSTR/MSE appears to be too large to have been selected at random from the appropriate F distribution.

Comparing the Variance Estimates: The *F* Test (2 of 2)

Sampling Distribution of MSTR/MSE



ANOVA Table for a Completely Randomized Design (1 of 3)

SST is partitioned into SSTR and SSE.

SST's degrees of freedom (df) are partitioned into SSTR's df and SSE's df.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	p-Value
Treatments	SSTR	k - 1	$\text{MSTR} = \dfrac{\text{SSTR}}{k-1}$	$\dfrac{\text{MSTR}}{\text{MSE}}$	
Error	SSE	n_T - k	$\text{MSE} = \dfrac{\text{SSE}}{n_T - k}$		
Total	SST	n_T - 1			

ANOVA Table for a Completely Randomized Design (2 of 3)

- SST divided by its degrees of freedom $n_T - 1$ is the overall sample variance that would be obtained if we treated the entire set of observations as one data set.
- With the entire data set as one sample, the formula for computing the total sum of squares, SST, is:

$$SST = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (x_{ij} - \bar{\bar{x}})^2 = SSTR + SSE$$

ANOVA Table for a Completely Randomized Design (3 of 3)

- ANOVA can be viewed as the process of partitioning the total sum of squares and the degrees of freedom into their corresponding sources: treatments and error.
- Dividing the sum of squares by the appropriate degrees of freedom provides the variance estimates, the F value and the *p*-value used to test the hypothesis of equal population means.

Test for the Equality of k Population Means (1 of 2)

- Hypotheses

$$H_0\colon \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$$

$$H_a\colon \text{Not all population means are equal.}$$

- Test Statistic

$$F = \frac{MSTR}{MSE}$$

Test for the Equality of k Population Means (2 of 2)

Rejection Rule:

p-value approach: Reject H_0 if the *p*-value $\leq \alpha$

Critical value approach: Reject H_0 if $F \geq F_\alpha$

Where the value of F_α is based on an F distribution with $k - 1$ numerator degrees of freedom and $n_T - k$ denominator degrees of freedom.

Testing for the Equality of k Population Means: A Completely Randomized Design (1 of 7)

AutoShine, Inc. is considering marketing a long-lasting car wax. Three different waxes (Type 1, Type 2, and Type 3) have been developed. In order to test the durability of these waxes, 5 new cars were waxed with Type 1, 5 with Type 2, and 5 with Type 3. Each car was then repeatedly run through an automatic carwash until the wax coating showed signs of deterioration.

The number of times each car went through the carwash before its wax deteriorated is shown on the next slide. AutoShine, Inc. must decide which wax to market. Are the three waxes equally effective?

Factor . . . Car wax

Treatments . . . Type 1, Type 2, Type 3

Experimental units . . . Cars

Response variable . . . Number of washes

Testing for the Equality of k Population Means: A Completely Randomized Design (2 of 7)

Observation	Wax Type 1	Wax Type 2	Wax Type 3
1	27	33	29
2	30	28	28
3	29	31	30
4	28	30	32
5	31	30	31
Sample Mean	29.0	30.4	30.0
Sample Variance	2.5	3.3	2.5

Testing for the Equality of k Population Means: A Completely Randomized Design (3 of 7)

$$H_0\colon \mu_1 = \mu_2 = \mu_3$$

H_a: Not all population means are equal.

Where:

μ_1 = the mean number of washes using Type 1 wax

μ_2 = the mean number of washes using Type 2 wax

μ_3 = the mean number of washes using Type 3 wax

Testing for the Equality of k Population Means: A Completely Randomized Design (4 of 7)

Mean Square Between Treatments: (Because the sample sizes are all equal)

$$\bar{\bar{x}} = \frac{(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)}{3} = \frac{(29 + 30.4 + 30)}{3} = 29.8$$

$$SSTR = 5(29 - 29.8)^2 + 5(30.4 - 29.8)^2 + 5(30 - 29.8)^2 = 5.2$$

$$MSTR = \frac{5.2}{(3 - 1)} = 2.6$$

Mean Square Error:

$$SSE = 4(2.5) + 4(3.3) + 4(2.5) = 33.2$$

$$MSE = \frac{33.2}{(15-3)} = 2.77$$

Testing for the Equality of k Population Means: A Completely Randomized Design (5 of 7)

Rejection Rule:

p-value approach: Reject H_0 if the *p*-value ≤ 0.05

Critical value approach: Reject H_0 if $F \geq 3.89$

Where the value of $F_{0.05} = 3.89$ is based on an F distribution with 2 numerator degrees of freedom and 12 denominator degrees of freedom.

Testing for the Equality of k Population Means: A Completely Randomized Design (6 of 7)

Test Statistic:

$$F = \frac{MSTR}{MSE} = \frac{2.60}{2.77} = 0.939$$

The *p*-value is greater than 0.10, where $F = 2.81$. Excel provides a *p*-value of 0.42. Therefore, we cannot reject H_0.

Conclusion:

There is insufficient evidence to conclude that the mean number of washes for the three wax types are not all the same.

Testing for the Equality of k Population Means: A Completely Randomized Design (7 of 7)

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	*F*	*p*-Value
Treatments	5.2	2	2.60	0.939	0.42
Error	33.2	12	2.77		
Total	38.4	14			

Testing for the Equality of k Population Means: An Observational Study

(1 of 8)

Example: Reed Manufacturing

Janet Reed would like to know if there is any significant difference in the mean number of hours worked per week for the department managers at her three manufacturing plants (in Buffalo, Pittsburgh, and Detroit). An F test will be conducted using $\alpha = 0.05$.

A simple random sample of five managers from each of the three plants was taken and the number of hours worked by each manager in the previous week is shown on the next slide.

Factor . . . Manufacturing plant

Treatments . . . Buffalo, Pittsburgh, Detroit

Experimental units . . . Managers

Response variable . . . Number of hours worked

Testing for the Equality of k Population Means: An Observational Study

(2 of 8)

Observation	Plant 1 Buffalo	Plant 2 Pittsburgh	Plant 3 Detroit
1	48	73	51
2	54	63	63
3	57	66	61
4	54	64	54
5	62	74	56
Sample Mean	55	68	57
Sample Variance	26.0	26.5	24.5

Testing for the Equality of k Population Means: An Observational Study

(3 of 8)

1. Develop the hypotheses.

$$H_0\colon \mu_1 = \mu_2 = \mu_3$$

H_a: Not all population means are equal.

Where:

μ_1 = the mean number of hours worked per week by the managers at Plant 1.

μ_2 = the mean number of hours worked per week by the managers at Plant 2.

μ_3 = the mean number of hours worked per week by the managers at Plant 3.

Testing for the Equality of k Population Means: An Observational Study

(4 of 8)

2. Specify the level of significance. α = 0.05
3. Compute the value of the test statistic.

Mean Square Due to Treatments (all sample sizes are equal):

$$\bar{\bar{x}} = \frac{(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)}{3} = \frac{(55 + 68 + 57)}{3} = 60$$

$$SSTR = 5(55 - 60)^2 + 5(68 - 60)^2 + 5(57 - 60)^2 = 490$$

$$MSTR = \frac{490}{(3 - 1)} = 245$$

Testing for the Equality of k Population Means: An Observational Study

3. Compute the value of the test statistic.

Mean Square Due to Error

$$SSE = 4(26) + 4(26.5) + 4(24.5) = 308$$

$$MSE = \frac{308}{(15-3)} = 25.667$$

$$F = \frac{MSTR}{MSE} = \frac{245}{25.667} = 9.55$$

Testing for the Equality of k Population Means: An Observational Study

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	p-Value
Treatment	490	2	245	9.55	.0033
Error	308	12	25.667		
Total	798	14			

Testing for the Equality of k Population Means: An Observational Study

(7 of 8)

p-value approach

4. Compute the *p*-value.

   With 2 numerator *df* and 12 denominator *df*, the *p*-value is 0.01 for F = 6.93.

   Therefore, the *p*-value is less than 0.01 for F = 9.55.

5. Determine whether to reject H_0. The *p*-value $\leq$ 0.05, so we reject H_0.

   We can conclude that the mean number of hours worked per week by department managers is not the same at all 3 plants.

Testing for the Equality of k Population Means: An Observational Study

(8 of 8)

Critical Value Approach

4. Determine the critical value and rejection rule.

Based on an F distribution with 2 numerator *df* and 12 denominator *df*, $F_{0.05} = 3.89$.
We will reject H_0 if $F \geq 3.89$.

5. Determine whether to reject H_0.

Because $F = 9.55 \geq 3.89$, we reject H_0.
We can conclude that the mean number of hours worked per week by department managers is not the same at all 3 plants.

Multiple Comparison Procedures

- Suppose that analysis of variance has provided statistical evidence to reject the null hypothesis of equal population means.
- Fisher's least significant difference (LSD) procedure can be used to determine where the differences occur.

Fisher's LSD Procedure (1 of 2)

Hypotheses:

$$H_0\colon \mu_i = \mu_j$$
$$H_a\colon \mu_i \neq \mu_j$$

Test Statistic:

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}}$$

Fisher's LSD Procedure (2 of 2)

Rejection Rule:

p-value approach: Reject H_0 if the *p*-value ≤ 0.05

Critical value approach: Reject H_0 if $t \leq -t_{\alpha/2}$ or $t \geq t_{\alpha/2}$

Where the value of $t_{\alpha/2}$ is based on a t distribution with $n_T - k$ degrees of freedom.

Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$ (1 of 6)

- Hypotheses:

$$H_0\colon \mu_i = \mu_j$$
$$H_a\colon \mu_i \neq \mu_j$$

- Test Statistic:

$$\bar{x}_i - \bar{x}_j$$

- Rejection Rule:

$$\text{Reject } H_0 \text{ if } \left|\bar{x}_i - \bar{x}_j\right| \geq \text{LSD}$$
$$\text{Where LSD} = t_{\alpha/2}\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$ (2 of 6)

Example: Reed Manufacturing

Recall that Janet Reed wants to know if there is any significant difference in the mean number of hours worked per week for the department managers at her three manufacturing plants.

Analysis of variance has provided statistical evidence to reject the null hypothesis of equal population means. Fisher's least significant difference (LSD) procedure can be used to determine where the differences occur.

Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$ (3 of 6)

For $\alpha = 0.05$ and $n_T - k = 15 - 3 = 12$ degrees of freedom $t_{0.025} = 2.179$.

$$\text{LSD} = t_{\alpha/2}\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

$$\text{LSD} = 2.179\sqrt{25.667\left(\frac{1}{5} + \frac{1}{5}\right)} = 6.98$$

Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$ (4 of 6)

LSD for Plants 1 and 2

- Hypotheses (A): $H_0\colon \mu_1 = \mu_2$

  $H_a\colon \mu_1 \neq \mu_2$

- Rejection Rule: Reject H_0 if $|\bar{x}_1 - \bar{x}_2| \geq 6.98$

- Test Statistic: $|\bar{x}_1 - \bar{x}_2| = |55 - 68| = 13$

Conclusion: The mean number of hours worked at Plant 1 is <u>not equal</u> to the mean number worked at Plant 2.

Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$ (5 of 6)

LSD for Plants 1 and 3

- Hypotheses (B): $H_0\colon \mu_1 = \mu_3$
  $H_a\colon \mu_1 \neq \mu_3$

- Rejection Rule: Reject H_0 if $|\bar{x}_1 - \bar{x}_3| \geq 6.98$

- Test Statistic: $|\bar{x}_1 - \bar{x}_3| = |55 - 57| = 2$

Conclusion: There is no significant difference between the mean number of hours worked at Plant 1 and the mean number of hours worked at Plant 3.

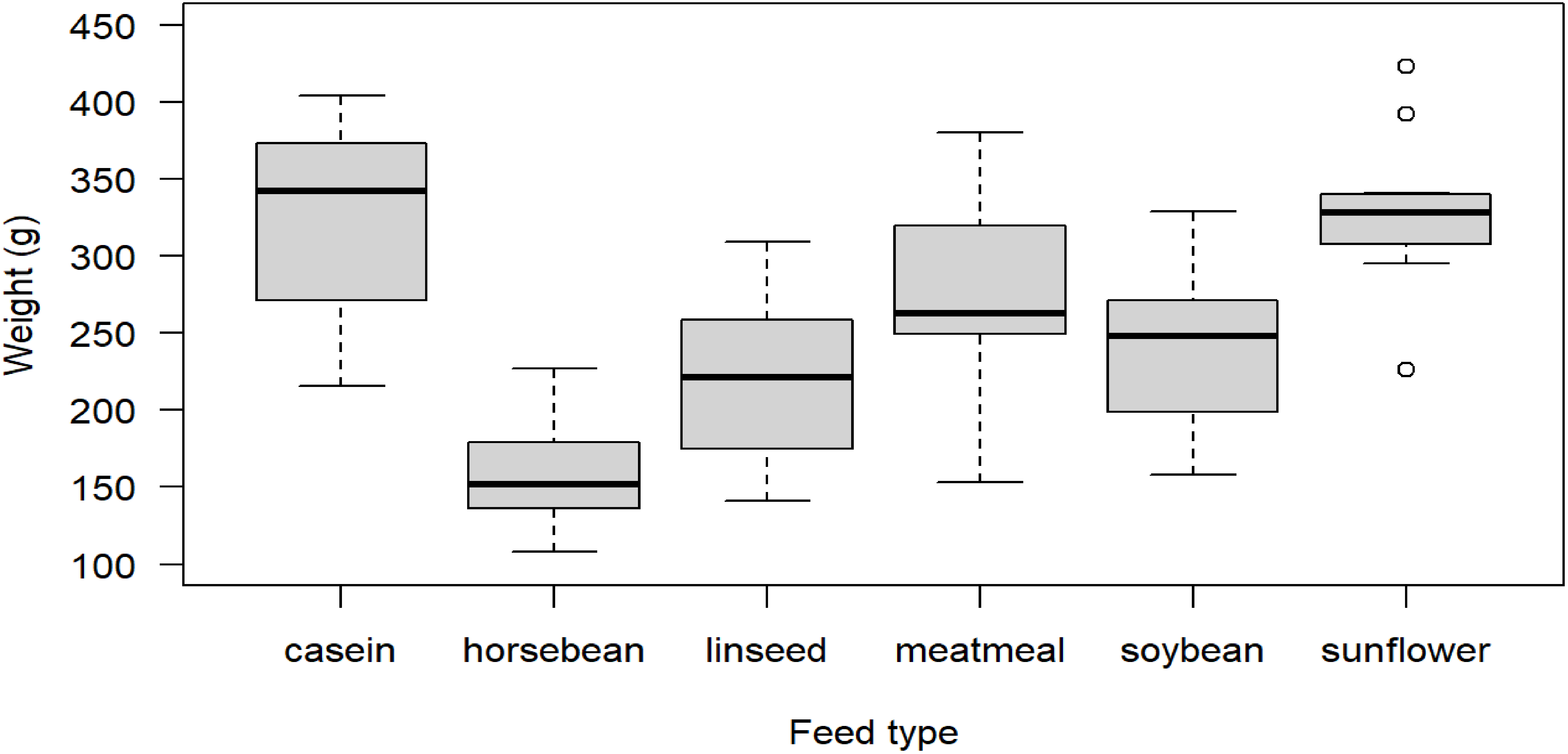
Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$ (6 of 6)

LSD for Plants 2 and 3

- Hypotheses (C): $H_0\colon \mu_2 = \mu_3$
  $H_a\colon \mu_2 \neq \mu_3$
- Rejection Rule: Reject H_0 if $|\bar{x}_2 - \bar{x}_3| \geq 6.98$
- Test Statistic: $|\bar{x}_2 - \bar{x}_3| = |68 - 57| = 11$

Conclusion: The mean number of hours worked at Plant 2 is <u>not equal</u> to the mean number worked at Plant

■ **R** data “chickwts”, weights of chickens fed six different types of feed.



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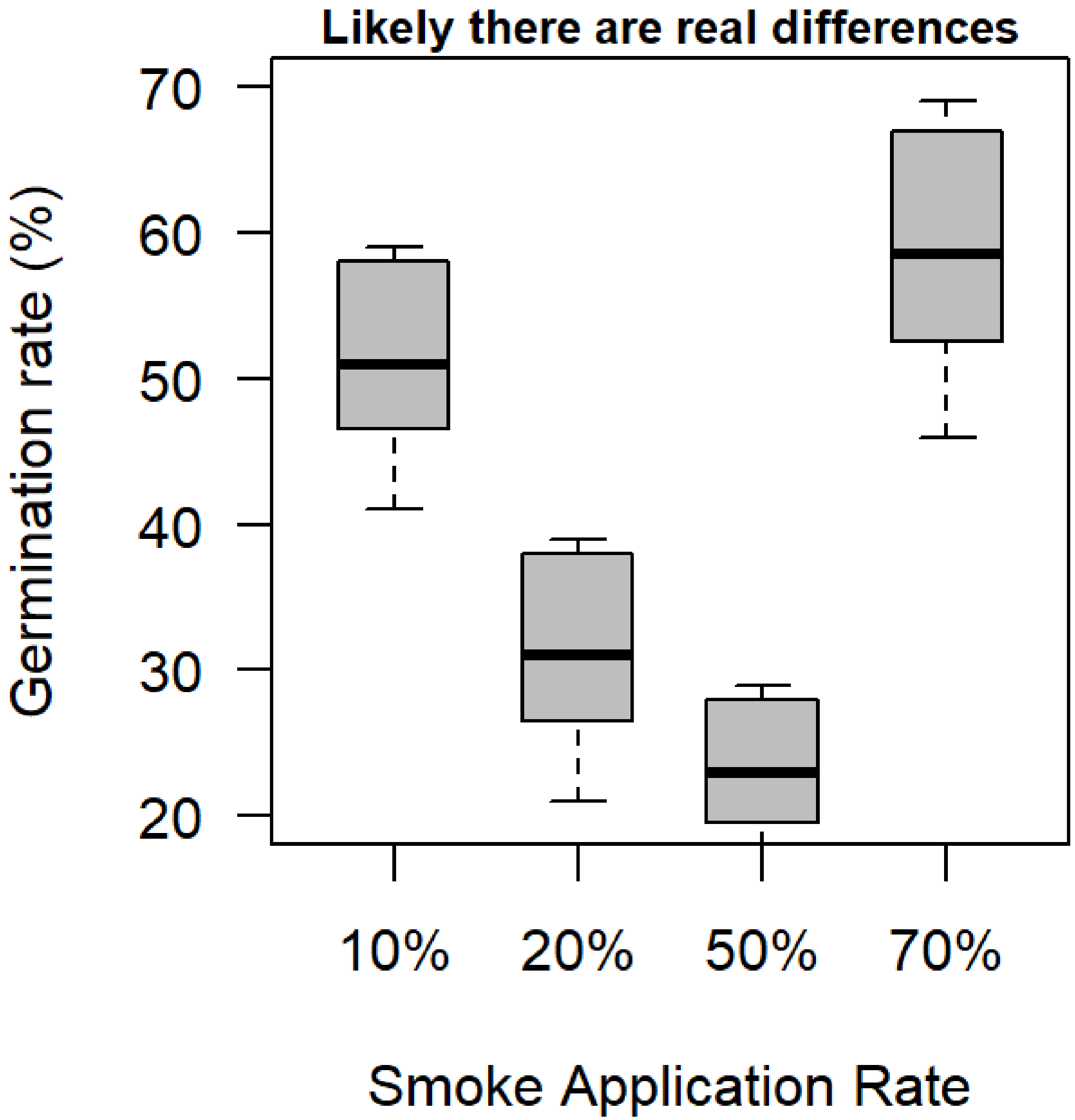
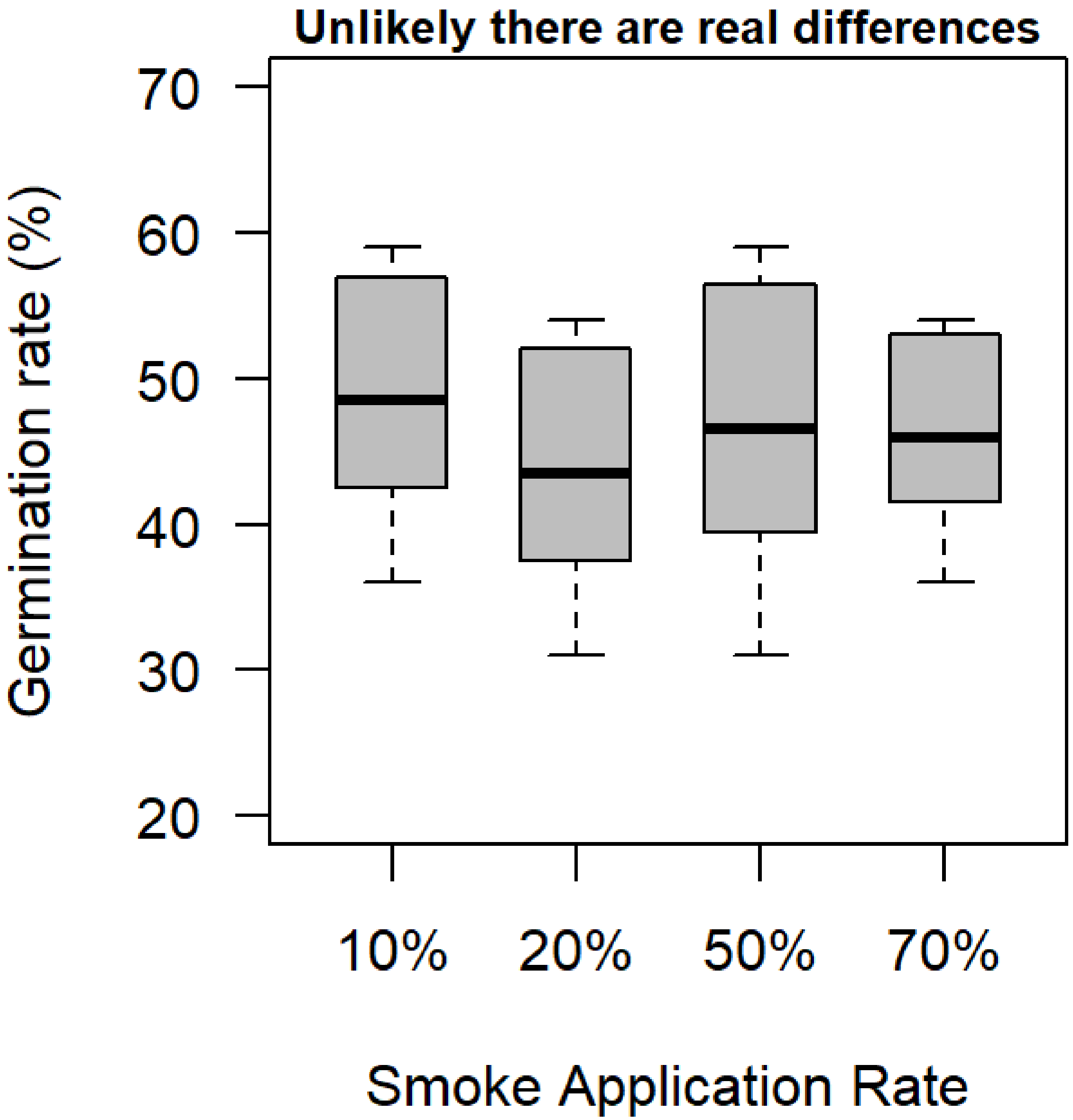
Pairwise t-test p-values: effect of feed on chicken weights

	Casein	Horsebean	Linseed	Meatmeal	Soybean
Horsebean	<0.001	-	-	-	-
Linseed	<0.001	0.094	-	-	-
Meatmeal	0.182	<0.001	0.094	-	-
Soybean	0.005	0.003	0.518	0.518	-
Sunflower	0.812	<0.001	<0.001	0.132	0.003

https://saeappliedstats.tech/analysis-of-variance.html

Boxplot can be useful in detecting between variation!





https://saeappliedstats.tech/analysis-of-variance.html

Type I Error Rates

The comparisonwise Type I error rate α indicates the level of significance associated with a single pairwise comparison.

The experimentwise Type I error rate α_{EW} is the probability of making a Type I error on at least one of the $(k-1)!$ pairwise comparisons.

$$\alpha_{EW} = 1 - (1 - \alpha)^{(k-1)!}$$

The experimentwise Type I error rate gets larger for problems with more populations (larger k).