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A Synthesis Mortality Model for the Elderly

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Mortality improvement has been a common phenomenon since the 20th century, and the human longevity continues to prolong. Postretirement life receives a lot of attention, and modeling mortality rates of the elderly (ages 65 years and beyond) is essential because life expectancy has reached the highest level in history. Mortality models can be divided into two groups, relational and stochastic models, but there is no consensus on which model is better in modeling mortality rates of the elderly. In this study, instead of choosing either a relational or stochastic model, we propose a synthesis model, selecting and modifying appropriate models from both groups, which not only has a satisfactory estimation result but also can be used for mortality projection. We use the data from the United States, the United Kingdom, Japan, and Taiwan (data were from the Human Mortality Database) to evaluate the proposed approach. We found that the proposed model performs well and is a possible choice for modeling mortality rates of the elderly.

1. INTRODUCTION

Mortality improvement has been well recognized by life insurers, pension providers, and government officials. As the number in the elder population increases, more related financial and insurance products are created, and suitable tools are needed to price these products. There is a growing interest in modeling and projecting mortality rates of the elderly (age 65 years and older), among all solutions in better managing longevity risk (Bohk-Ewald and Rau 2017; Li et al. 2011; Preston and Stokes 2012; Thatcher et al. 1999). However, the process of modeling mortality rates of elderly population has not worked well yet in Taiwan, because of a lack of quantity and good quality in the data on the elderly. For example, Taiwan has an excellent population registration system but the highest attained age of 100+ (ages 100 and beyond) is found in records from 1992 forward. This indicates that only 25 years of data are available if we want to study the mortality rates of the oldest-old (85+) population.

The inadequate data availability for the elderly population increases the difficulty of mortality modeling. There is no consensus about the future trend of elderly mortality, and whether life expectancy has a limit is one of the points in dispute (Carnes et al. 2003; Yue 2012). Researchers and institutions also noticed that mortality improvement patterns of different age groups may not be the same (Bohk-Ewald and Rau 2017; Börger and Schupp 2018; Kannisto 1994; Kannisto et al. 1994). Figure 1 displays the trends of mortality rates for those at age 30 years from the United States, the United Kingdom, Japan, and Taiwan for the period 1970–2005, whereas Figure 2 shows those for persons ages 65 and 80 at the same period. While mortality improvements of younger populations have begun to stabilize, there are still obvious decreasing trends for mortality rates of the elderly. As a result, applying stochastic mortality models such as the Lee–Carter model (Lee and Carter 1992), assuming that the changing rates with respect to time of all ages are similar, may not result in the required accuracy for mortality fitting or projection. We need to modify the mortality models in order to incorporate various rates of mortality improvement at different ages.

Mortality models can be separated into two categories: the traditional relational models and the modern stochastic models. Traditional relational models, such as the Gompertz model, the Makeham model, and the Weibull model, often use single-year mortality data and assume that the age-specific mortality rates satisfy a certain function form. The Coale–Kisker model (Coale and Kisker 1990) and the logistic model are two other popular relational models that receive lots of attention in modeling elderly mortality. Most relational models usually provide sound mortality estimation, but they are not good for prediction

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FIGURE 1. Trend of Mortality Rates of Persons at Age 30 for Various Countries (1970–2005).

over time. Many researchers proposed modifications to increase the predicting accuracy of relational models. Tsai and Yang (2014) proposed a linear regression approach to relate one mortality sequence to another of equal length. Cadena and Denuit (2016) applied the accelerated failure time model and propose a semiparametric accelerated hazard relational model.

Stochastic models, on the other hand, use multiple-year data and focus on the time/cohort trend of mortality rates. Most of them are good for mortality prediction but may not be used for data extrapolation, that is, predicting the mortality rates beyond the highest attained ages. The LC model (Lee and Carter 1992) is probably the most widely used stochastic model in mortality predictions and applications. The RH model (Renshaw and Haberman 2006) generalized the LC model by including a cohort effect. The CBD model (Cairns et al. 2006), has been commonly used for modeling the mortality rates of higher ages (Cairns et al. 2009). Among these three models, only the CBD model can extrapolate the mortality rates of these ages without mortality records.

More studies have focused on the stochastic models in recent years, mainly because of their performance in modeling mortality improvement, and many modifications of the LC and CBD models are proposed, including, to name a few, those of Chen and Cox (2009), Haberman and Renshaw (2009), Li et al. (2009), Cox et al. (2010), Cairns et al. (2009), and Mitchell et al. (2013). To improve predictability, Brouhns et al. (2002) substituted a log-bilinear Poisson regression model for single vector decomposition (SVD) in the LC model's parameter estimation. Ahcan et al. (2013) suggested forecasting mortality for small populations by mixing appropriately the mortality data obtained from other populations. Tsai and Lin (2017) incorporated Bühlmann credibility into mortality models to improve forecasting performances. However, as mentioned previously, most of the analysis results of stochastic models are limited to the age range of the historical data. For ages beyond the given sample age range, stochastic models may not provide results for mortality estimation and projection.

The main goal of this study is to propose a mortality model for post-retirement populations, ages 65 years and older. Instead of setting up a new model, we aim to combine existing mortality models with accurate and stable results in mortality estimation and/or prediction. In particular, we first choose a stochastic model as the basis and modify it with a relational model. We hope that the proposed synthesis model can be used to extrapolate mortality rates for higher ages, as well as possessing good performance in estimation and prediction. In this study, we choose the LC and CBD models for the group of stochastic models and use the Gompertz, Coale-Kisker, and logistic models for the group of relational model.

The concept of combining two different models is used quite often in constructing life tables (or mortality graduation), particularly for graduating the mortality rates for the elderly. For example, it is believed that the quality of data on the elderly is more reliable in data from the Medicare program, with the U.S. Social Security Administration, and the mortality rates for ages 85 and over were produced based on the Medicare data. In constructing the 1979–1981 U.S. Life Table, the mortality rates for ages 85 and over were decided by the following formula (Brown 1993):

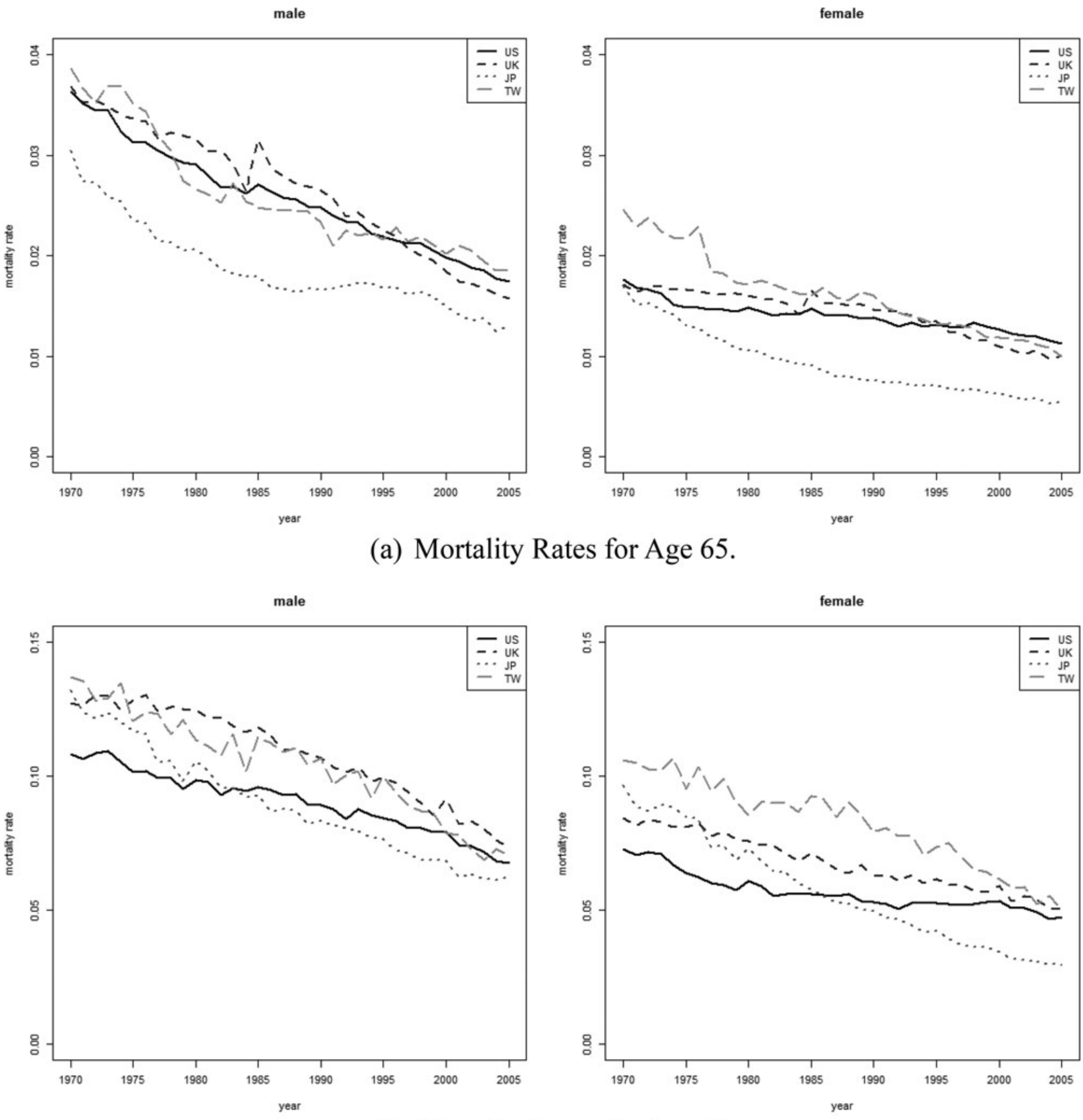

FIGURE 2. Trends of Mortality Rates for Ages 65 and 80 Years for Various Countries (1970–2005).

$$q_x = \frac{1}{11}\left[(95-x)q_x^C + (x-84)q_x^M\right]. \tag{1}$$

where q_x^C and q_x^M are the estimated mortality rates of age $x(85 \le x \le 94)$ from the population census and Medicare, respectively, and higher weights are on the Medicare mortality rates for higher ages. This simple but useful weighted formula provides smooth and reliable estimates between two models, provided that they produce smooth and accurate estimates and the weights are properly chosen. This formula also encourages the proposed synthesis model.

There are other methods for handling the case of populations with limited data, including survivor ratio method and partial standard mortality ratio (SMR) method (Lee 2003; Thatcher et al. 2002; Wang et al. 2018). Terblanche (2016) proposed

TABLE 1
List of Mortality Models Used in This Study

Model	Target variable	Assumption
LC	Central death rate	$\ln m_{x,t} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$
CBD	Mortality rate	$\text{logit}(q_{x,t}) = \ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k_{C1,t} + k_{C2,t}(x-\bar{x}) + \varepsilon_{x,t},$
Gompertz	Force of mortality	$\mu_x = BC^x$
CK	Central death rate	$\min_{\alpha_K, s_K} \sum_x w_x \left[\ln m_x - \alpha_K - k_{K,85}(x-84) + \frac{(x-84)(x-85)}{2} s_K\right]^2$
Logistic	Mortality rate	$\eta_x = -\ln p_x = \frac{\exp[\beta_{L,0}+\beta_{L,1}\cdot(x+0.5)]}{1+\exp[\beta_{L,0}+\beta_{L,1}\cdot(x+0.5)]}$

retrospective tests based on a number of extrapolative methods to forecast mortality rates of elderly populations in Australia. For a population of 1 million or more, graduation methods such as the Whittaker and LC models can produce stable mortality estimates at ages other than the elderly (Wang et al. 2018). We often apply relational models to acquire mortality estimates for the elderly. For example, the Taiwan government uses a two-stage estimation to construct official life tables.[1] At the first stage, the Whittaker method is used to graduate mortality rates for ages 0–79 years and the Gompertz model is applied to acquire mortality rates for ages 65 years and over. At the second stage, the mortality rates at ages 65–79 years are determined by the linear combination of Whittaker and Gompertz estimates; higher weights are on the Gompertz estimates for higher ages.

This article is organized as follows: Section 2 provides a brief review of both relational models and stochastic model applied in the article. Section 3 describes the process of constructing the synthesis models. In Section 4, we compare the proposed models with existing models and apply the forecasted mortality rates to calculate the premium of a pure annuity. Finally, the conclusion and discussion are given in Section 5.

2. REVIEW OF MORTALITY MODELS

We first give a brief introduction of mortality models used in this study (Table 1), following by the description of proposed approach. For the group of stochastic models, we choose the LC and the CBD models. Both are popular and are well known for providing good mortality estimation and prediction. For the group of relational models, we selected the Gompertz, the Coale–Kisker, and the logistic models. These models are known for providing sound fitting results for mortality rates for the elderly's We introduce the stochastic models first.

Among all stochastic mortality models, the simplicity and accuracy of the LC model make it the most popular and widely used. Lee and Carter (1992) assumed that the central death rate of an individual aged x at year t follows the form

$$\ln m_{x,t} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}, \tag{2}$$

where α_x describes the average age pattern of mortality over time, β_x is the deviations from the average pattern, κ_t describes the variation in the level of mortality over time, and $\varepsilon_{x,t}$ is the error term.

The parameters are subject to constraints

$$\sum \kappa_t = 0 \text{ and } \sum \beta_x = 1, \tag{3}$$

to ensure model identification. When forecasting mortality rates, it is assumed that the α_x's and β_x's remain constant over time and the values of κ_t are modeled by random walk with drift:

$$\kappa_t = \kappa_{t-1} + \phi + e_t \tag{4}$$

[1]*Source:* Ministry of Interior, Department of Statistics, http://www.moi.gov.tw/stat

where $e_t \sim N(0, {\sigma_{LC}}^2)$, and φ is known as the drift parameter. The parameters can be estimated via singular value decomposition (SVD), weighted least squares, the maximum likelihood estimation, or an approximation method, if there are missing data.

The CBD model, proposed by Cairns et al. (2006), was designed to model mortality rates for higher ages. Assuming that there are two time trends, $k_{C1,t}$ and $k_{C2,t}$, the model, for given age range $[x_1, x_1 + n - 1]$ and time period $[t_1, t_1 + K - 1]$, is given by

$$\text{logit}(q_{x,t}) = \ln\left(\frac{q_{x,t}}{1 - q_{x,t}}\right) = k_{C1,t} + k_{C2,t}(x - \bar{x}) + \varepsilon_{x,t}, \tag{5}$$

where

$$\bar{x} = \frac{\sum_{x=x_1}^{x_1+n-1} x}{n} \tag{6}$$

and $q_{x,t}$ is the mortality rate of an individual aged x in year t.

The errors are assumed to be independent and identically distributed and the two time trends are modeled by a bivariate random walk with drift:

$$\begin{cases} k_{C1,t} = k_{C1,t-1} + \phi_{C1} + e_{C1,t} \\ k_{C2,t} = k_{C2,t-1} + \phi_{C2} + e_{C2,t} \end{cases}. \tag{7}$$

The Gompertz law probably is a well-known relational model and is often applied to acquire mortality rates for the elderly. The Gompertz law of mortality assumes that the force of mortality at age x takes the form of $\mu_x = BC^x$, where $B>0$ and $C>1$. Under this assumption, the probability that an individual at age x would survive to age $x+1$, denoted by p_x, is

$$p_x = \exp\left[-BC^x(C-1)/\ln C\right]. \tag{8}$$

Let l_x be the number of individuals alive at age x; then the amount who will survive 1 year later is

$$l_{x+1} = l_x \cdot \exp\left[-BC^x(C-1)/\ln C\right]. \tag{9}$$

From Equation (8), we also have

$$\ln p_{x+1}/\ln p_x = C. \tag{10}$$

Therefore, if the mortality pattern of the observed population follows the Gompertz law, the ratios of $\ln p_{x+1}$ to $\ln p_x$ remain fairly constant.

The CK model (Coale and Kisker 1990) assumes that the central death rate can be modeled as

$$m_x = m_{x-1} \cdot \exp\left[k_{K,85} + (x-85)\cdot s\right], \quad \text{for } x \geq 85, \tag{11}$$

where

$$s = -\frac{\ln\left(\frac{m_{84}}{m_{110}}\right) + 26k_{K,85}}{325} \quad \text{and} \quad k_{K,85} = \ln\left(\frac{m_{85}}{m_{84}}\right). \tag{12}$$

Given that m_{110} is set at 1.0/0.8 for men/women, the mortality rates above age 85 can be obtained recursively and allow us to carry out the extrapolation of the age-extended mortality table. A further revised method used weighed least squares on the following quadratic form related to the central death rate:

$$\min_{\alpha_K, s_K} \sum_x w_x \left[\ln m_x - \alpha_K - k_{K,85}(x-84) + \frac{(x-84)(x-85)}{2} s_K\right]^2 \tag{13}$$

The logistic model was proposed by Perks (1932), who found empirically that the values of force of mortality in a life table that he was examining could be fitted by a logistic function (Thatcher et al. 1999). In other words, the population growth rate declines with population numbers and reaches a limit. In this article, we assume that the logarithm of p_x, the probability that an individual at age x would survive to age $x+1$, is modeled as the following:

$$\eta_x = -\ln p_x = \frac{\exp\left[\beta_{L,0} + \beta_{L,1} \cdot (x+0.5)\right]}{1 + \exp\left[\beta_{L,0} + \beta_{L,1} \cdot (x+0.5)\right]}. \tag{14}$$

3. THE PROPOSED SYNTHESIS MODEL

Before introducing the process of combining two mortality models, we note that mortality models may not use the same target variables, and these can be death rate, central death rate, or force of mortality. The Gompertz model is based on the assumption of the force of mortality, whereas the LC model is based on the central death rate and the CBD model is based on death rates. Thus, we need to modify all the models to have the same target variable. By definition, the central death rate of a person aged x at year t, denoted by $m_{x,t}$, is calculated as follows:

$$m_{x,t} = \frac{d_{x,t}}{L_{x,t}} = \frac{d_{x,t}}{\int_0^1 l_{x+y,t+y-1} dy}, \tag{15}$$

where $d_{x,t} = l_{x,t} - l_{x+1,t+1}$ is the number of deaths between age x and age $x+1$ at year t, and $L_{x,t}$ is the exposure, or number of person-years lived between age x and $x+1$ at year t. We first assume that the force of mortality follows the Gompertz assumption, $\mu_{x,t} = B_t C_t^x$, at year t.[2] Assuming that the force of mortality is uniformly distributed within year t, we further approximate the exposure (or stationary population) $\mathrm{L}_{x,t}$ of age x at time t, under the uniform distribution of death, via

$$L_{x,t} \cong \frac{l_{x,t} + l_{x+1,t}}{2} = \frac{l_{x,t}(1 + p_{x,t})}{2}. \tag{16}$$

Then we have

$$m_{x,t} = \frac{d_{x,t}}{L_{x,t}} = \frac{2(1 - p_{x,t})}{(1 + p_{x,t})}. \tag{17}$$

Given that $p_{x,t} = \exp\left[-B_t C_t^x (C_t - 1)/\ln C_t\right]$ from the Gompertz model, we can derive the following estimation for the central death rate:

$$\ln\left(-\ln\left(\frac{1 - \frac{m_{x,t}}{2}}{1 + \frac{m_{x,t}}{2}}\right)\right) = \ln\left(B_t (C_t - 1)/\ln C_t\right) + x \ln C_t. \tag{18}$$

We combine the advantages of both the relational and the stochastic mortality models to build the proposed model. The process of constructing the synthesis mortality model is similar to that of constructing life tables in Taiwan. The relational models have a better fit for the mortality rates of the elderly but may not have good estimation for all ages generally. This is the reason for using the Gompertz law for the mortality rates at only ages 65 years and over in Taiwan. Thus, we use the stochastic models for younger ages, apply relational models for the older ages, and design a method to connect these two models in between. In other words, we use the two different mortality models before and after a chosen age. The chosen age, termed the transition age (Figure 3), is set so that the morality curve will be separated into two parts: the stochastic model before the transition age and the synthesis model after that. Figure 4 briefly describes the process of the proposed synthesis model. The estimation of the stochastic model is performed first, followed by that of the relational model for mortality rates of ages over the transition age.

The model combining the LC model and the Gompertz law is called the LC-G model. Similarly, the one combining the LC model and the CK/logistic model is referred as the LC-CK/LC-Log model, while that of combining the CBD model and the Gompertz law, the CK model, and the logistic model are referred as the CBD-G, the CBD-CK, and the CBD-Log model, respectively.

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[2]Notice that the Gompertz parameters now have subscripts t. Instead of setting a fixed Gompertz parameter for the whole period, we estimate the parameter on a year-to-year basis.

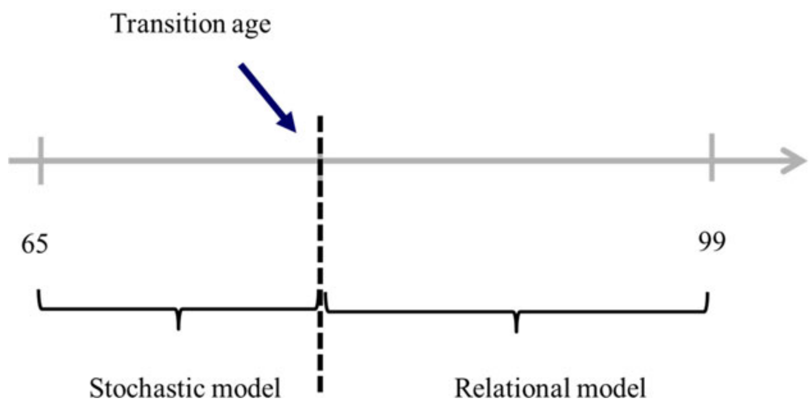

FIGURE 3. The Transition Age After Which We Apply the Synthesis Process.

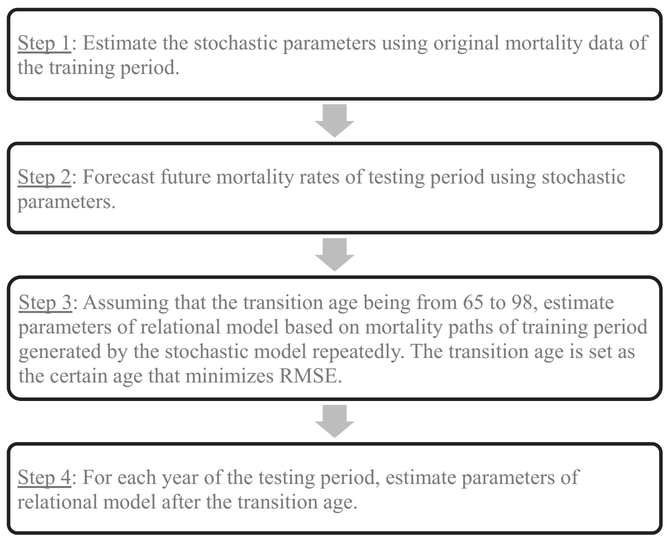

FIGURE 4. The Synthesis Process.

4. EMPIRICAL STUDY AND APPLICATIONS

4.1. Empirical Data and Determining the Transaction Ages

We use empirical data to evaluate the proposed synthesis model described in the previous section. The mortality data used are from four countries: the United States, the United Kingdom, Japan, and Taiwan (source: Human Mortality Database [HMD]), with age range 65–99 years and time period 1970–2009. We use the backcasting (or cross-validation) procedure as mentioned by Dowd et al. (2010), Tsai and Yang (2015), and Tsai and Lin (2017). Since projection results are highly sensitive to calibration periods and accuracy does not consistently increase with increased length of fitting period (Van Berkum et al. 2016; Terblanche 2016), we consider the backcasting procedure with a 10-year training period and a 5-year testing period. We assume that the training period is $[t_0, t_0 + 9]$, where t_0 is 1970, 1975, 1980, 1985, 1990, and 1995, with 5 years of overlap between two consecutive periods, and the testing period is $[t_0 + 10, t_0 + 14]$. The data from training periods are used to construct mortality models and the fitted model is applied to the data from the testing periods.

We first look at the results of the transition age. As defined, the certain transition age point is set so that the morality curve will be separated into two parts: the stochastic model before the transition age, and the combined model after that. To find the transition age for each year period $[t_0: t_0 + 9]$, we assume the transition ages as 65 to 98 years and repeatedly run the synthesis process for each, as described in Figure 4. The resulted transition age is then set as the one that minimizes the root mean square error.

TABLE 2
Transition Ages for the Synthesis Models

	LC + CK				LC + Gompertz			
Data period	United States	United Kingdom	Japan	Taiwan	United States	United Kingdom	Japan	Taiwan
1970–1979	77	77	67	85	78	65	75	70
1975–1984	77	77	73	96	77	65	75	71
1980–1989	77	69	76	96	77	68	76	75
1985–1994	75	68	74	71	77	75	77	74
1990–1999	75	68	75	77	77	65	75	66
1995–2004	66	66	75	74	66	66	75	74
	CBD + CK				CBD + Gompertz			
Data period	United States	United Kingdom	Japan	Taiwan	United States	United Kingdom	Japan	Taiwan
1970–1979	66	72	98	96	65	86	98	98
1975–1984	66	75	95	97	65	81	97	98
1980–1989	66	75	95	96	65	77	98	98
1985–1994	66	66	98	95	65	67	98	98
1990–1999	66	66	69	98	65	69	68	98
1995–2004	66	66	66	98	65	69	65	98

Table 2 summarizes the transition ages of synthesis models for countries at different data periods.[3] Generally speaking, most transition ages lie in the age range of 65–80 years, except for the cases of Taiwan and Japan. It seems that the trend of mortality rates of the elderly is different from that for younger ages, and introducing relational models for the older ages may improve the model fit.

The results of CBD model and Gompertz model require further attention, especially for the combination of CBD model and Gompertz model. For this treatment combination, the transition age is 65 years in the case of U.S. data and is 98 years in the case of Taiwan data (and some of the Japan data). This indicates that the Gompertz model has better fit than the CBD model in the case of American elderly persons, but the CBD model fits better than the Gompertz model in the case of Taiwanese elderly persons (i.e., there is no need to consider the Gompertz model). Since the CBD and Gompertz models are designed to model the elderly's mortality rates, our results imply that the fitting performance of elderly mortality models is likely to be data dependent, which is mentioned in the previous studies (e.g., Wang et al. 2016). The combination of CBD and the Coale–Kisker model also reveals similar pattern regarding the determination of the transition age.

Note that in addition to applying mortality models directly, we can also check whether the empirical data support using these models. As mentioned in the previous section, the ratios of $\ln p_{x+1}$ to $\ln p_x$ should remain fairly constant if the mortality pattern of the observed population follows the Gompertz law. The results show that we cannot reject the assumption that the mortality rates of elderly population follow the Gompertz law. Yue (2002) proposed using a bootstrap simulation to construct the confidence interval of parameter C, and we can use this to verify whether the elderly mortality rates follow the Gompertz assumption. The bootstrap simulation (Appendix A) suggests that the Gompertz model is a feasible assumption for the data from four countries, since the intersections of confidence intervals are not empty. We can consider similar tests for other mortality models as part of the exploratory data analysis before constructing the synthesis model. We use the empirical data to evaluate the proposed synthesis model in the next section.

4.2. Fitting and Forecasting Results

We can evaluate the performance of mortality models based on the training data (estimation) or the testing data (prediction). We consider the case of training data first. More parameters in the model usually would have smaller estimation error, but there is risk from using too many parameters (i.e., overfitting). Thus, we use the Akaike information criterion (AIC) to evaluate the model fit for the training data. The AIC was proposed by Akaike (1973) and has been widely used for model selection. It is defined as

[3]We did not apply the transition age process on the Lee-Log and the CBD-Log models because estimating parameters for the logistic model requires regression on all ages.

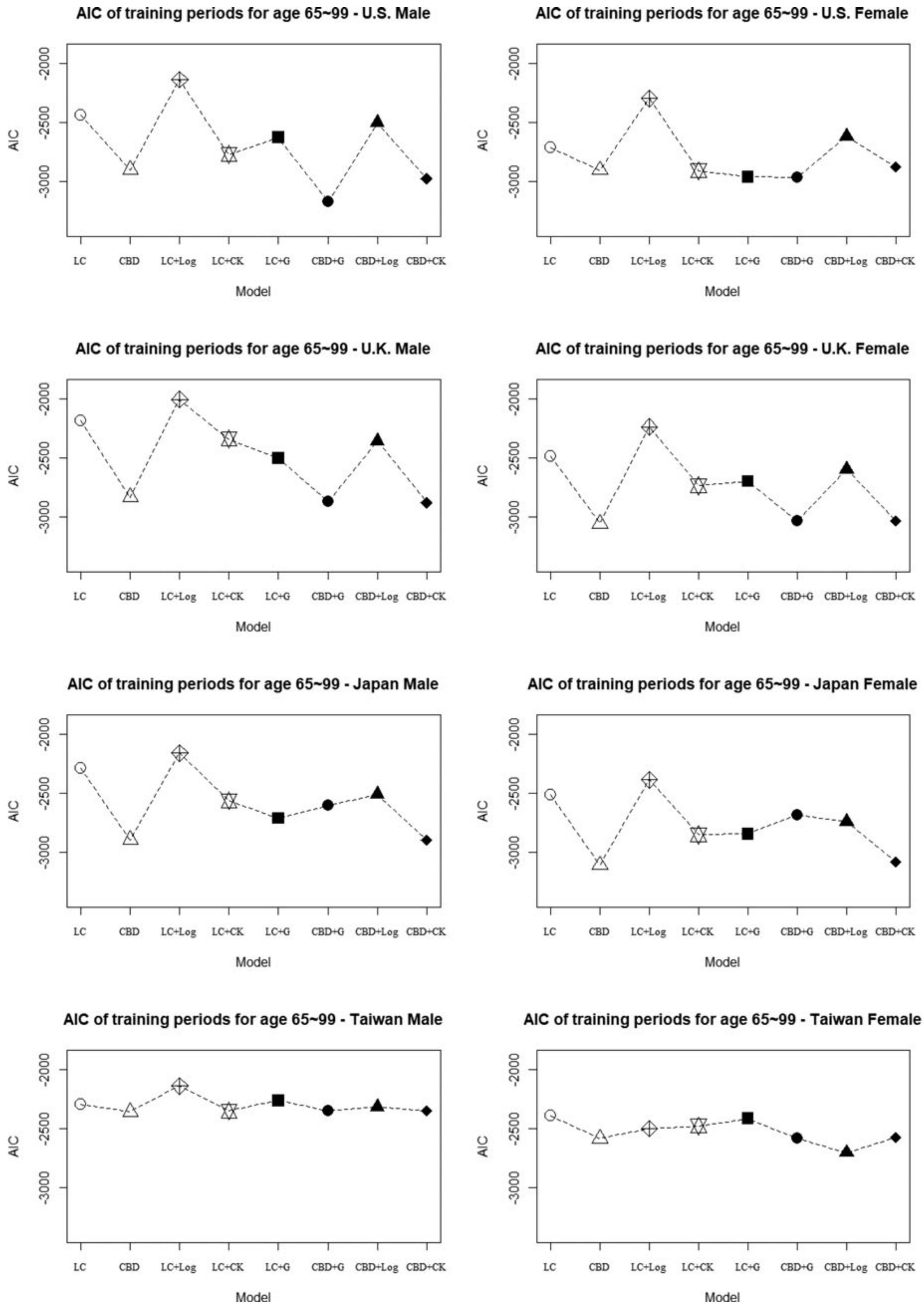

FIGURE 5. The AIC Values for Each Country Using 1970–2009 Data.

TABLE 3
AIC Results for All Models

Country	Model	Male	Female
United States	LC	–2442.39	–2710.06
	CBD	–2900.39	–2901.45
	LC + logistic	–2137.96	–2298.65
	LC + CK	–2770.56	–2909.07
	LC + Gompertz	–2627.22	–2960.20
	CBD + Gompertz	–3170.57	–2962.67
	CBD + logistic	–2505.06	–2616.81
	CBD + CK	–2976.2	–2874.85
United Kingdom	LC	–2188.61	–2485.58
	CBD	–2832.27	–3054.74
	LC + logistic	–2006.08	–2238.35
	LC + CK	–2340.22	–2735.48
	LC + Gompertz	–2502.70	–2699.24
	CBD + Gompertz	–2871.12	–3032.61
	CBD + logistic	–2358.40	–2598.59
	CBD + CK	–2877.96	–3033.13
Japan	LC	–2287.47	–2511.17
	CBD	–2894.01	–3106.61
	LC + logistic	–2162.06	–2387.90
	LC + CK	–2563.81	–2850.71
	LC + Gompertz	–2711.83	–2844.52
	CBD + Gompertz	–2602.32	–2683.34
	CBD + logistic	–2511.96	–2740.09
	CBD + CK	–2897.29	–3078.81
Taiwan	LC	–2292.74	–2391.17
	CBD	–2357.67	–2584.49
	LC + logistic	–2141.92	–2502.43
	LC + CK	–2351.51	–2479.17
	LC + Gompertz	–2265.03	–2415.86
	CBD + Gompertz	–2353.67	–2580.49
	CBD + logistic	–2318.17	–2703.30
	CBD + CK	–2348.34	–2575.16

Note: The numbers with shadow are the best among all models.

$$AIC = -2\log(ML) + 2K, \tag{19}$$

where ML is the maximum likelihood under the given model and K is the number of fitted parameters. If the errors of mortality models are assumed to be normally distributed with constant variance, then AIC can be also computed from the least squares regression (Burnham and Anderson 2002), redefining the AIC as

$$AIC = N\log\left(\frac{RSS}{N}\right) + 2K, \tag{20}$$

where RSS is the residual sum of squares,

$$RSS = \sum_{j=1}^{T}\sum_{i=0}^{n-1}\left(\hat{q}_{x+i,j} - q_{x+i,j}\right)^2, \tag{21}$$

TABLE 4
Fitting Outperformance Scores of All Models

	RMSE		MAPE		
Model	65–84	85–99	65–84	85–99	Total
LC	9	1	15	2	27
CBD	9	5	4	7	25
LC + logistic	1	2	2	2	7
LC + CK	0	8	4	5	17
LC + Gompertz	4	7	9	5	25
CBD + Gompertz	13	8	8	11	40
CBD + logistic	2	3	0	3	8
CBD + CK	10	14	6	13	43
Total score	48	48	48	48	192

Note: The numbers with shadow are the best among all models.

TABLE 5
Forecasting Outperformance Scores of All Models

	RMSE		MAPE		
Model	65–84	85–99	65–84	85–99	Total
LC	4	0	5	2	11
CBD	9	4	9	6	28
LC + logistic	7	4	8	4	23
LC + CK	0	2	1	1	4
LC + Gompertz	2	10	5	11	28
CBD + Gompertz	9	10	12	10	41
CBD + logistic	9	6	2	2	19
CBD + CK	8	12	6	12	38
Total score	48	48	48	48	192

Note: The numbers with shadow are the best among all models.

where N is the number of observations and K is the total number of estimated regression parameters. Smaller AIC values indicate better performance in estimation. Given that we use the weighted least squares to estimate parameters of our combined models, we applied Equation (20) for the calculation of the AIC values of all models, as shown in Figure 5. In general, the synthesis models tend to have smaller AIC values, compared to those without combining the relational models, although additional parameters are added. For example, the LC + CK model has smaller AIC values than the LC model. However, no models have the smallest AIC values in all cases. Detailed results are given in Table 3.

Of course, we can use the estimation errors to evaluate the mortality models, such as in terms of root mean square error (RMSE) and mean absolute percentage error (MAPE), which are defined as

$$RMSE = \frac{1}{T}\sum_{j=1}^{T}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(\hat{q}_{x+i,j}-q_{x+i,j}\right)^2\right]^{1/2} \tag{22}$$

and

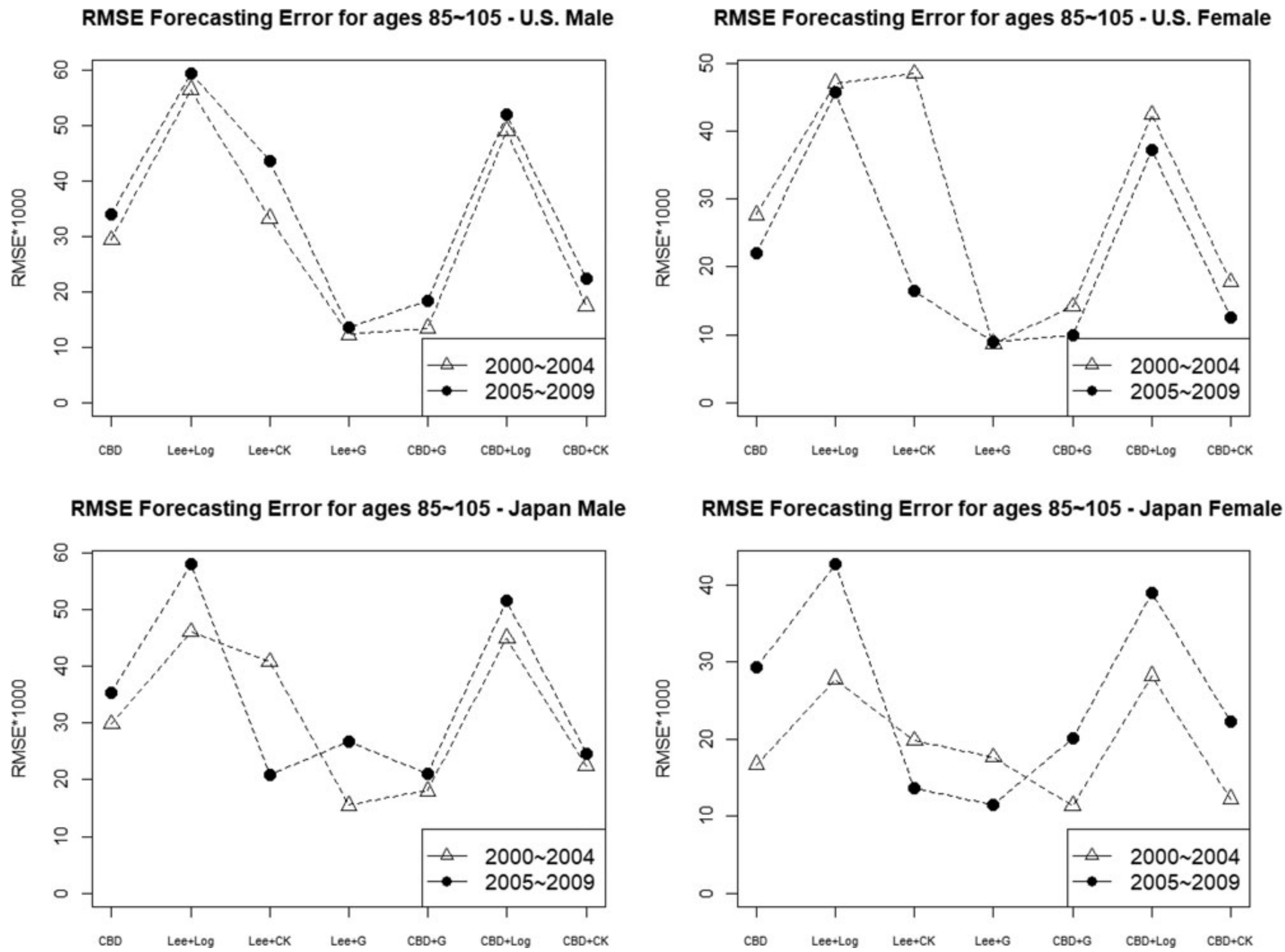

FIGURE 6. RMSE Forecasting Error for Ages 85–105 Years.

$$MAPE = \frac{1}{T}\sum_{j=1}^{T}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left|\hat{q}_{x+i,j} - q_{x+i,j}\right|\right], \tag{23}$$

where T is the time period, n is the age range, $\hat{q}_{x+i,j}$ is the observation value, and $q_{x+i,j}$ is the estimated value.

We divide the results into two age groups: [65, 84] and [85, 99], for younger and older elderly mortality. Detailed results are in Appendix B-1 and B-2. In terms of estimation errors, the synthesis models have smaller errors for the age group [85, 99], but the LC and CBD models have smaller errors for the group [65, 84]. The estimation errors generally are larger for the group [85, 99], almost 10 times as large as those for the group [65, 84].

It seems that the fitting performance of models is data dependent and we cannot find a single model that dominates the others. Still, we look for the mortality models that have fitted well in most countries, and we apply the voting method to determine this. The voting is to count the scores for each data set, and the model thaty has the best performance gets 3 points and the second and third best get 2 points and 1 point, respectively. Outperformance scores of overall fitting are based on RMSE and MAPE (Table 4). Given that the total outperformance score is 48 points for each comparison, with 4 treatment combinations, models with scores over 32 points can be considered as having a performance better than average. The CBD + Gompertz and CBD + CK models are the best among all models.

We can follow the same procedure to evaluate the forecasting performance. We apply the models acquired from the training data to the testing data. Again, we use the RMSE and MAPE as the criteria for comparison, and the calculation results are shown in Appendix Tables B-3 and B-4. Unlike the results of fitting errors, the synthesis models have smaller forecasting errors, but no mortality models can outperform other models. Thus, we also use the voting method for overall comparison (Table 5). Again, the CBD + Gompertz and CBD + CK models are the best among these with respect to forecasting errors. It

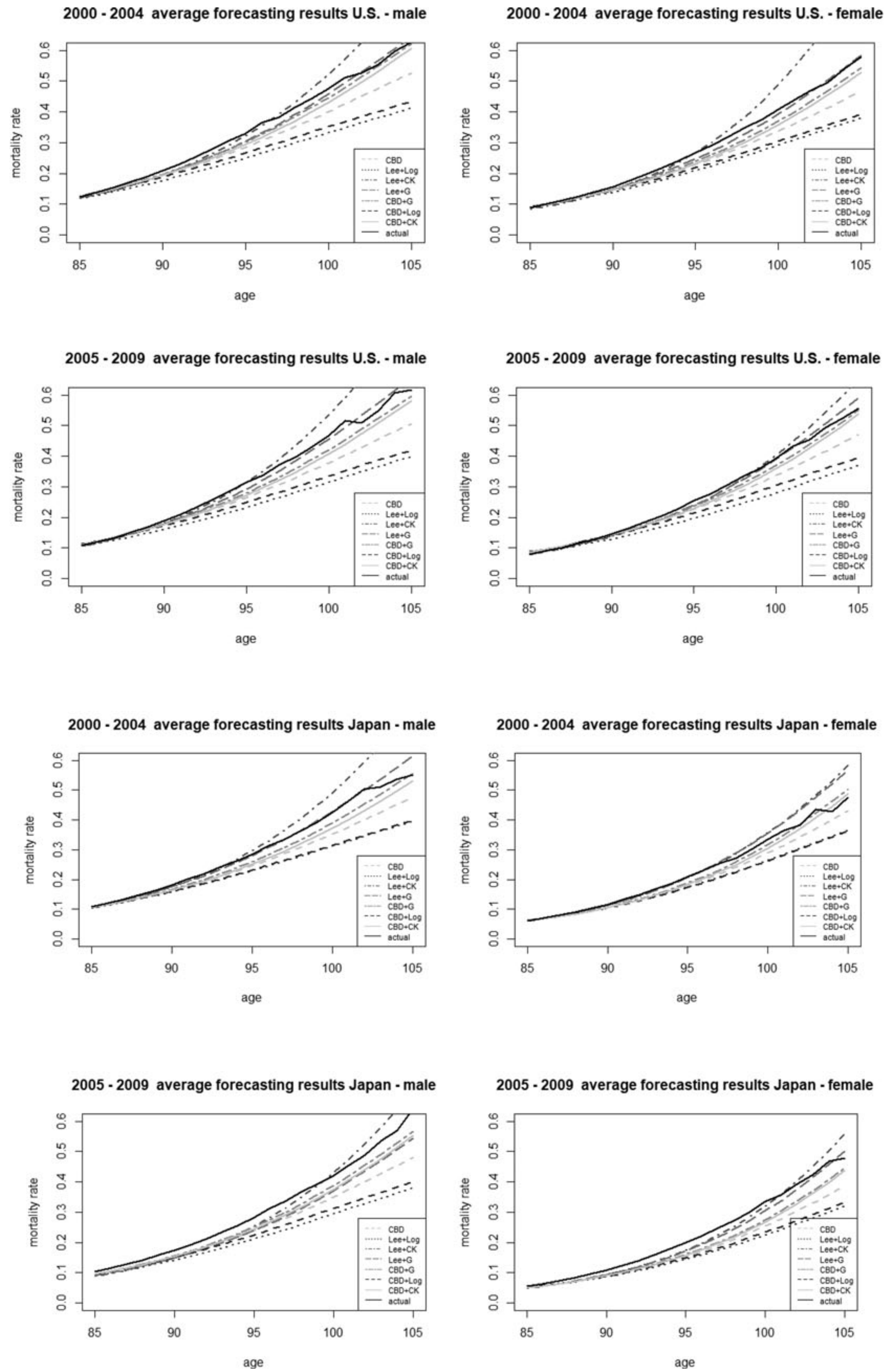

FIGURE 7. Five-Year Average Mortality Curve for Ages 85–105 Years (CBD and Synthesis Models). *Note:* The black solid lines are the observed mortality rates.

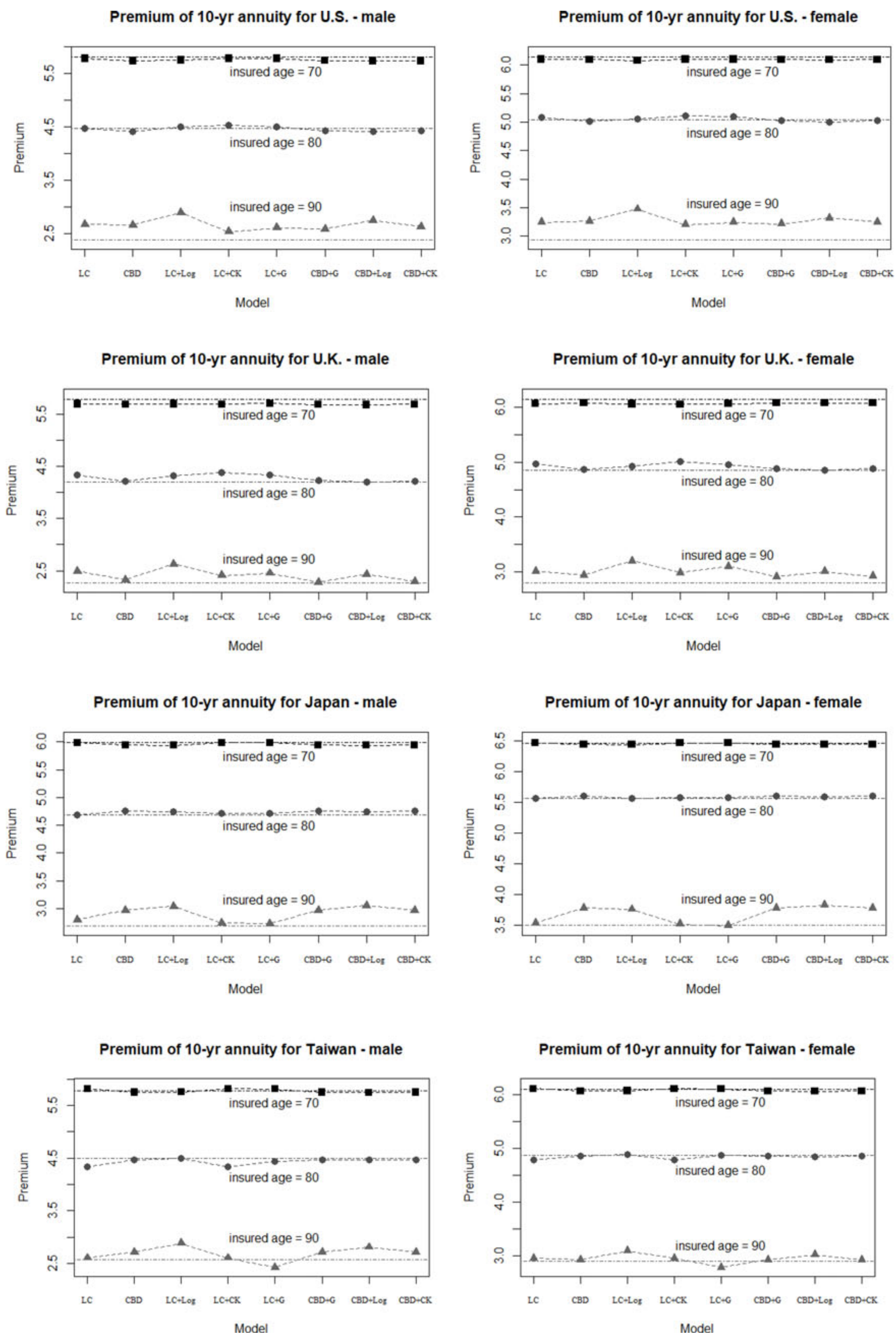

FIGURE 8. Single Premium of 10-Year Pure Annuity for Insured Ages 70, 80, and 90 Years.

should be noted that, among all other models, the LC and CBD models have fine performance in fitting for ages 65–84, whereas the CBD + Gompertz and CBD + CK models perform well in fitting and forecasting for the oldest-old (ages 85–99).

One of our motivations for introducing a synthesis model is extrapolation beyond the highest recorded ages, and thus we illustrate the extension of mortality rates for age 100 and over. Because not all countries have data beyond age 100, we select the United States and Japan to demonstrate our approach. Also, due to the data availability, only two data periods are considered: [1990, 1999] and [1995, 2004] for training periods, [2000, 2004] and [2005, 2009] for testing periods. We estimate parameters of mortality models from ages 65 to 99 years, and apply the acquired models to ages 100 to 105 years. Note that the LC model is not considered here since it cannot be used to extrapolate.

Figure 6 displays the forecasting results of our synthesis models, compared with the CBD model, and Figure 7 shows the average mortality curves of ages 85–105 years. We use the RMSE errors to demonstrate the forecasting results. In general, the CBD + Gompertz and LC + Gompertz models have the smallest forecasting errors in almost all cases, and the average errors are smaller than 30%, which is not bad for ages 85–105. Also, as shown in Figure 6, the combined model CBD + Gompertz always has a smaller RMSE than the CBD model for all 8 cases, which is equivalent to p value $= (0.5)^8 \approx 0.004$ (i.e., run test). This indicates the CBD + Gompertz model is better than the CBD model for the group of ages 85+ years. It seems that combining the stochastic and relational models is a feasible approach. Also, it seems that the CBD model may overestimate mortality improvements for ages over 95, resulting in the lower mortality curve shown in Figure 7.

4.3. Pricing Annuity Products

Other than the empirical study, we also consider applications by applying our synthesis model to annuity products. To simplify the discussion, we consider the single premium of a 10-year life annuity product that pays $1 at the end of each year. The interest rate, i, is 2% and the insured ages are 70, 80, and 90 years. We use the data from 1980–1999 for parameter estimation, and the premiums are calculated for the insured period 2000–2009 based on the acquired mortality models. We also calculate the actual values of the annuities based on the real mortality data of the insured period. Detailed results of premium calculations and estimated errors (reported as % of actual values) are listed in Appendix C.

Figure 8 shows the annuity premiums derived from all models, compared with the actual values based on the true mortality rates. Not surprisingly, the premiums of mortality models with lower forecasting errors (the proposed approaches) in the previous section are closer to the actual values. This means that the synthesis models can be used in practice. However, all mortality models underestimate the premiums for insured age 70 and overestimate the amount for insured age 90. This indicates that all models tend to overestimate the mortality rates at younger ages and underestimate those at older ages.

5. CONCLUSION AND DISCUSSIONS

Postretirement life has received a lot of attention in recent years. Forecasting the life expectancy for the elderly is essential to deal with the longevity risk, and mortality models are a popular choice. However, there is no consensus on which model is better in modeling the mortality rates of the elderly. Also, due to the data availability, it is often necessary to interpolate mortality rates beyond the highest recorded ages. In this article, we propose a synthesis model, combining relational and stochastic mortality models, in order to achieve a better fit of mortality rates for the elderly and mortality extrapolation.

The idea behind the synthesis model is similar to the construction of life tables in Taiwan, where two different approaches are used: one for the elderly and the other for younger ages. Basically, we think that stochastic models are preferable for younger ages and relational models can be used for the elderly. In this study, we consider LC and CBD models for younger ages and Gompertz, CK, and logistic models for the older ages. We use empirical data from the Human Mortality Database (the United States, the United Kingdom, Japan, and Taiwan; years 1970–2009) to evaluate the proposed approach. The data are separated into training and testing periods, and the RMSE and MAPE are used as error criteria. The synthesis mortality models generally have smaller fitting errors (and AIC values) in the training periods and smaller forecasting errors in the testing periods.

In addition, we found that the LC and CBD models perform well for mortality rates of ages 65 to 84 years and the synthesis models are a good choice for older ages. The synthesis models also have stability performance for mortality extrapolation, up to age 105 years. It seems that the synthesis model performs especially well for the elderly and the oldest-old (i.e., ages 85 and over), and it can be used for annuity products and insurance products related to older ages. However, similar to the results of previous studies, we found that no mortality models can dominate other models at all cases and the fitting results are highly data dependent. We only include four countries in this study, and we should consider more data in the future.

There are many choices for the stochastic and relational models. In addition to using the historical data and the backcasting procedure to choose the feasible combination of models, we can also use the idea of exploratory data analysis to screen possible choices of models. We can adapt the idea of Yue (2002) and develop tools for checking whether the mortality data satisfy certain model assumption. For example, bootstrapping simulation can be used to construct possible range of parameter C for the Gompertz model. This can help to narrow down the possible choices of mortality models.

There is a small difference between the synthesis model and life table construction in Taiwan. We use one single connection (single transition age) between two models, while Taiwan's life tables use a transition period (15 to 20 ages). The advantage of using a transition period is that the mortality rates are smoother around the connect points and this won't create jumps at consecutive ages. This is important in calculating insurance premiums. Also, since no mortality models can dominate, we should impose possible variations (or variance) to the mortality predictions. One possible approach is to use the block bootstrap simulation and derive 1,000 or 10,000 sample paths of predictions. The variance can be calculated via these sample paths. Another possibility is to adapt a Bayesian approach via Markov-chain Monte Carlo (MCMC) method, but it would require more computations.

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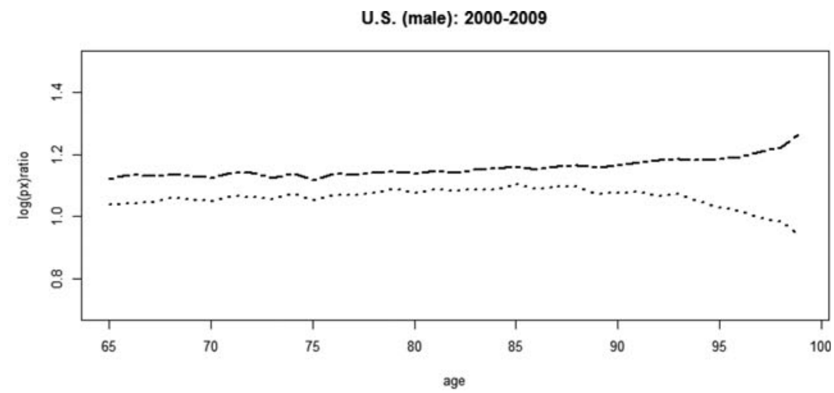
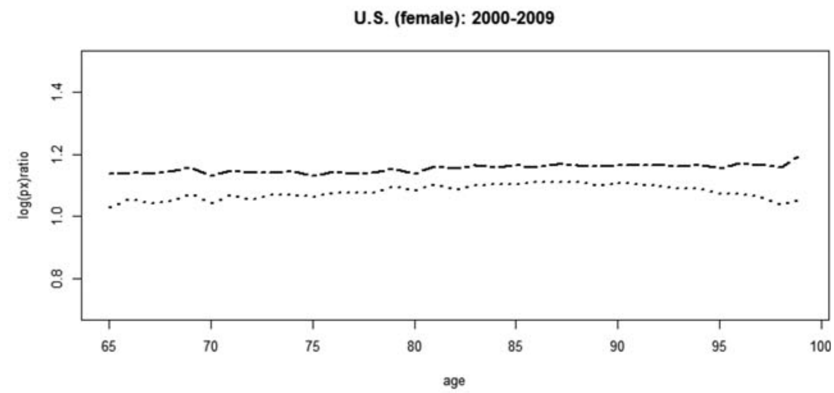
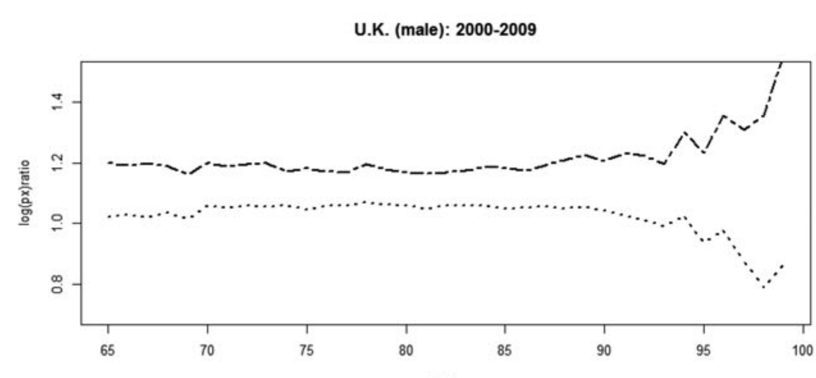
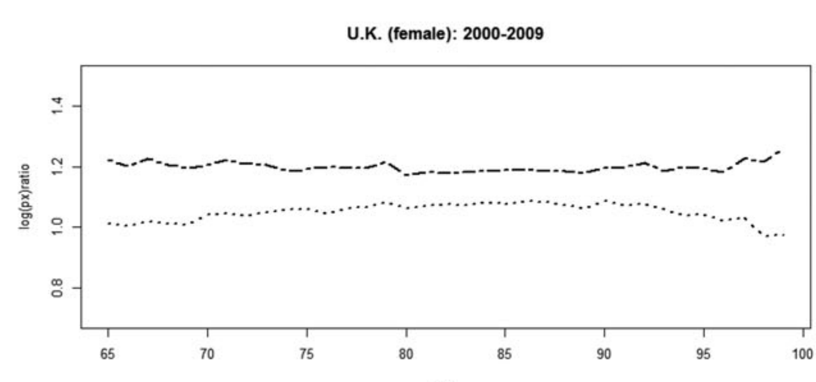
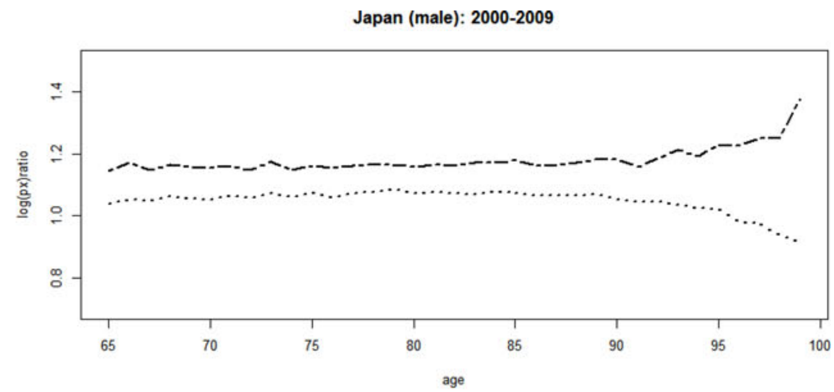
APPENDIX A. BOOTSTRAP CONFIDENCE INTERVALS OF UNITED STATES, UNITED KINGDOM, JAPAN, AND TAIWAN

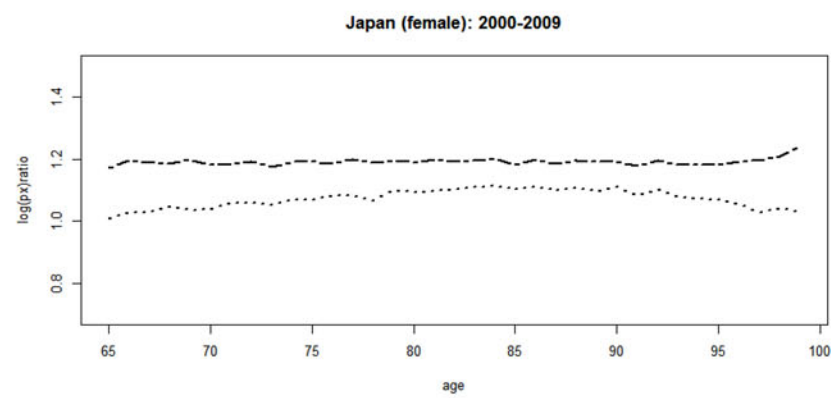


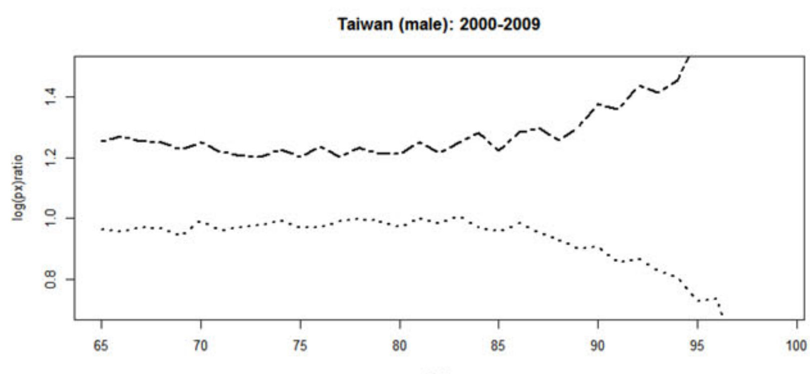


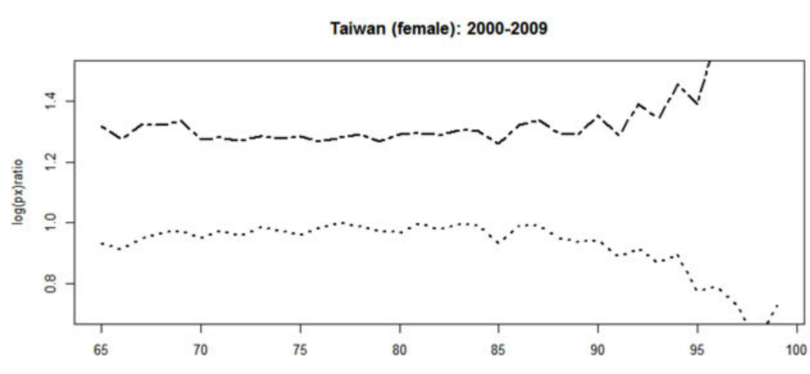





Japan (male): 2000-2009
log(px)ratio
age


Japan (female): 2000-2009
log(px)ratio
age


Taiwan (male): 2000-2009
log(px)ratio
age


Taiwan (female): 2000-2009
log(px)ratio
age

APPENDIX B. EMPIRICAL RESULTS OF MORTALITY MODELS

TABLE B.1
Fitting Errors for Ages 65–84 Years

		Male		Female	
Country	Model	RMSE ($\times 10^{-3}$)	MAPE (%)	RMSE ($\times 10^{-3}$)	MAPE (%)
United States	LC	3.3560	3.1277	1.4594	2.1621
	CBD	2.1185	2.5550	1.8546	4.0348
	LC + logistic	3.5496	3.2161	1.7389	3.4426
	LC + CK	4.6253	3.4927	2.4752	2.7939
	LC + Gompertz	3.9362	3.2945	1.7601	2.2613
	CBD + Gompertz	1.9626	2.3263	1.6819	3.6132
	CBD + logistic	2.3927	2.9576	2.1067	4.5994
	CBD + CK	2.0941	2.4997	1.7936	3.8961
United Kingdom	LC	4.9721	3.8591	2.1960	2.6960
	CBD	2.4688	2.3131	1.8309	3.1763
	LC + logistic	4.6858	3.7752	2.2085	2.7896
	LC + CK	5.8437	4.3659	2.8245	3.0330
	LC + Gompertz	5.2510	4.3488	2.6865	2.9121
	CBD + Gompertz	2.4403	2.3125	1.7862	3.0816
	CBD + logistic	2.6401	2.5159	2.0874	3.7577
	CBD + CK	2.4272	2.2817	1.8207	3.1184
Japan	LC	3.5715	3.0268	1.7504	2.1575
	CBD	2.1805	2.7498	1.6379	3.7064
	LC + logistic	4.1610	3.5400	2.4842	3.7494
	LC + CK	4.3507	3.3342	2.5940	2.6477
	LC + Gompertz	4.3719	3.2891	2.5311	2.5368
	CBD + Gompertz	2.1934	2.6369	1.6582	3.4820
	CBD + logistic	2.2652	3.0344	1.6669	4.0903
	CBD + CK	2.2044	2.7541	1.6462	3.7030
Taiwan	LC	5.8133	4.5638	3.9634	4.1601
	CBD	6.8117	5.7656	4.4357	5.1083
	LC + Logistic	7.8068	6.1683	4.9867	5.3945
	LC + CK	6.9567	5.1280	4.6875	4.6720
	LC + Gompertz	6.4367	4.7221	4.1599	4.2192
	CBD + Gompertz	6.8117	5.7656	4.4357	5.1083
	CBD + Logistic	6.6875	5.7677	4.4041	5.1468
	CBD + CK	6.8117	5.7656	4.4357	5.1083

Note: The numbers with shadow are the best among all models.

TABLE B.2
Fitting Errors for Ages 85–99 Years

Country	Model	Male		Female	
		RMSE ($\times 10^{-3}$)	MAPE (%)	RMSE ($\times 10^{-3}$)	MAPE (%)
United States	LC	37.4976	11.3087	25.5738	9.0608
	CBD	24.1130	6.6857	24.2132	8.4894
	LC + logistic	55.4611	16.6398	44.2728	16.0002
	LC + CK	19.7779	7.7307	16.8179	7.3884
	LC + Gompertz	24.6415	8.7118	15.4925	6.9827
	CBD + Gompertz	13.3335	3.8946	17.7827	6.4567
	CBD + logistic	39.9940	10.4867	34.0051	11.1086
	CBD + CK	17.7228	5.3219	20.6657	7.6032
United Kingdom	LC	53.8685	13.1828	35.3625	10.6348
	CBD	24.8964	4.7524	18.2539	5.5562
	LC + logistic	66.3180	15.4857	47.6979	14.6451
	LC + CK	35.8483	9.6428	20.8625	7.8345
	LC + Gompertz	27.8738	7.1356	21.7907	8.1852
	CBD + Gompertz	20.6717	4.2787	16.2918	4.5904
	CBD + logistic	47.8889	9.1208	33.8142	9.1015
	CBD + CK	19.8658	3.8724	15.4734	4.9204
Japan	LC	47.3160	12.2084	35.0886	10.2843
	CBD	23.1915	5.7631	17.1989	5.6509
	LC + logistic	52.6172	14.1613	38.1043	13.0589
	LC + CK	28.1403	8.3231	18.0771	6.6864
	LC + Gompertz	22.8317	7.0423	18.7452	6.3077
	CBD + Gompertz	35.8247	8.1371	32.9483	8.0402
	CBD + logistic	38.1077	8.6331	26.763	8.2957
	CBD + CK	21.6431	5.3908	16.8237	5.4409
Taiwan	LC	44.5132	12.4697	38.6652	10.8262
	CBD	57.7218	20.0147	37.3526	10.9785
	LC + logistic	51.8190	16.1094	30.8100	8.8296
	LC + CK	38.1068	11.5223	31.9804	9.5293
	LC + Gompertz	42.0679	12.6552	32.4046	9.8854
	CBD + Gompertz	57.7218	20.0147	37.3526	10.9785
	CBD + logistic	50.7823	16.6938	28.6391	8.1733
	CBD + CK	57.7215	20.0147	37.3525	10.9785

Note: The numbers with shadow are the best among all models.

TABLE B.3
Forecasting Errors for Ages 65–84 Years

		Male		Female	
Country	Model	RMSE ($\times 10^{-3}$)	MAPE (%)	RMSE ($\times 10^{-3}$)	MAPE (%)
United States	LC	3.3740	3.9212	2.0057	3.8829
	CBD	2.8898	3.9629	2.6157	5.6883
	LC + logistic	3.7241	4.1513	2.4206	5.1069
	LC + CK	4.6562	4.4027	3.0765	4.7087
	LC + Gompertz	3.8357	4.0913	2.1757	3.9097
	CBD + Gompertz	2.7170	3.6410	2.4714	5.2919
	CBD + logistic	3.1676	4.4455	2.8240	6.2341
	CBD + CK	2.8342	3.8672	2.5631	5.5667
United Kingdom	LC	4.0196	4.2840	2.2153	4.4132
	CBD	3.0639	3.8845	2.2554	4.5885
	LC + logistic	3.8397	3.9780	2.2655	4.2714
	LC + CK	4.9212	5.0706	2.4642	4.4244
	LC + Gompertz	4.2132	4.3535	2.3816	4.1642
	CBD + Gompertz	2.8290	3.7237	2.0652	4.4468
	CBD + logistic	3.5034	4.3308	2.6572	5.2325
	CBD + CK	2.9109	3.8051	2.1679	4.5415
Japan	LC	4.0920	6.1659	1.8233	4.8791
	CBD	2.6074	4.5133	1.5196	4.4374
	LC + logistic	4.6493	6.1616	2.3510	5.4622
	LC + CK	4.7505	6.4669	2.3772	5.1434
	LC + Gompertz	4.7055	6.3933	2.3313	5.1169
	CBD + Gompertz	2.6725	4.5106	1.5464	4.2312
	CBD + logistic	2.5333	4.5737	1.5036	4.7498
	CBD + CK	2.6360	4.5406	1.5280	4.4394
Taiwan	LC	7.3002	8.2912	5.3172	7.7119
	CBD	5.9859	6.1255	4.2835	5.9536
	LC + logistic	5.8909	6.0399	4.0245	5.2832
	LC + CK	7.4214	8.1515	5.2624	7.4556
	LC + Gompertz	7.1015	8.1854	5.1306	7.6294
	CBD + Gompertz	5.9859	6.1255	4.2835	5.9536
	CBD + logistic	5.9436	6.2005	4.3752	6.1213
	CBD + CK	5.9859	6.1255	4.2835	5.9536

Note: The numbers with shadow are the best among all models.

TABLE B.4
Forecasting Errors for Ages 85–99 Years

Country	Model	Male		Female	
		RMSE ($\times 10^{-3}$)	MAPE (%)	RMSE ($\times 10^{-3}$)	MAPE (%)
United States	LC	39.4826	11.5883	27.6544	9.9862
	CBD	29.7181	8.0761	30.3576	10.7362
	LC + logistic	62.4814	18.1527	50.6321	17.8374
	LC + CK	22.2683	8.1030	21.6274	9.1569
	LC + Gompertz	26.1881	9.0437	18.5894	8.2137
	CBD + Gompertz	16.6971	5.0162	21.5568	8.4009
	CBD + logistic	46.0261	11.7647	39.9645	13.1683
	CBD + CK	22.2178	6.5447	25.6593	9.6790
United Kingdom	LC	51.9006	12.8568	34.7497	10.6434
	CBD	22.6302	5.0615	19.2496	6.0992
	LC + logistic	61.9280	15.0403	48.4727	14.9654
	LC + CK	52.0008	11.9610	23.1984	8.6991
	LC + Gompertz	27.7059	7.5214	26.6892	9.6011
	CBD + Gompertz	20.2961	4.8595	16.0250	5.1120
	CBD + logistic	42.6221	8.7252	33.1332	9.3357
	CBD + CK	18.3238	4.3022	16.9487	5.5190
Japan	LC	45.2495	12.9741	31.2740	10.4128
	CBD	24.9037	7.4609	18.5358	7.7907
	LC + logistic	51.7791	15.7953	36.8234	14.5451
	LC + CK	30.0134	9.6770	23.1293	8.9763
	LC + Gompertz	22.3085	8.0699	18.1496	7.1200
	CBD + Gompertz	30.9035	8.0922	26.5387	8.5839
	CBD + logistic	38.4739	10.5045	28.0938	10.5815
	CBD + CK	22.7956	7.0297	17.3811	7.5203
Taiwan	LC	47.9116	12.6169	41.1403	11.4193
	CBD	45.2508	14.9925	35.6224	12.2041
	LC + logistic	40.1348	12.6390	27.9482	9.2390
	LC + CK	44.5323	12.7162	38.2318	11.2188
	LC + Gompertz	40.9803	11.2288	31.7637	9.9403
	CBD + Gompertz	45.2508	14.9925	35.6224	12.2041
	CBD + logistic	39.9630	12.8570	27.8872	9.9199
	CBD + CK	45.2506	14.9925	35.6231	12.2042

Note: The numbers with shadow are the best among all models.

APPENDIX C

TABLE C.1
Single Premium of 10-Year Pure Annuity Paying $1 Each Year—United States

	Male			Female		
Model	70	80	90	70	80	90
Actual cost of annuity	5.8109	4.4646	2.3788	6.1381	5.0381	2.9376
LC	5.7842	4.4748	2.6695	6.1014	5.0846	3.2451
	–0.46%	0.23%	12.22%	–0.60%	0.92%	10.47%
CBD	5.7420	4.4193	2.6552	6.0971	5.0165	3.2645
	–1.19%	–1.01%	11.62%	–0.67%	–0.43%	11.13%
LC + Log	5.7526	4.5020	2.8967	6.0812	5.0576	3.4755
	–1.00%	0.84%	21.77%	–0.93%	0.39%	18.31%
LC + CK	5.7816	4.5371	2.5336	6.0998	5.1172	3.2039
	–0.50%	1.63%	6.51%	–0.62%	1.57%	9.06%
LC + G	5.7825	4.5059	2.6051	6.1013	5.0917	3.2441
	–0.49%	0.93%	9.51%	–0.60%	1.07%	10.43%
CBD + G	5.7462	4.4259	2.5790	6.0992	5.0265	3.2149
	–1.11%	–0.86%	8.42%	–0.63%	–0.23%	9.44%
CBD + Log	5.7369	4.4083	2.7410	6.0946	5.0031	3.3172
	–1.27%	–1.26%	15.23%	–0.71%	–0.69%	12.92%
CBD + CK	5.7418	4.4279	2.6255	6.0964	5.0244	3.2481
	–1.19%	–0.82%	10.37%	–0.68%	–0.27%	10.57%

TABLE C.2
Single Premium of 10-Year Pure Annuity Paying $1 Each Year—United Kingdom

	Male			Female		
Model	70	80	90	70	80	90
Actual cost of annuity	5.7830	4.2021	2.2652	6.1471	4.8601	2.7991
LC	5.6945	4.3331	2.4933	6.0703	4.9700	3.0157
	–1.53%	3.12%	10.07%	–1.25%	2.26%	7.74%
CBD	5.6866	4.2074	2.3276	6.0856	4.8734	2.9406
	–1.67%	0.13%	2.76%	–1.00%	0.27%	5.05%
LC + Log	5.6955	4.3147	2.6242	6.0648	4.9259	3.1992
	–1.51%	2.68%	15.85%	–1.34%	1.35%	14.29%
LC + CK	5.6867	4.3849	2.4093	6.0615	5.0144	2.9784
	–1.67%	4.35%	6.36%	–1.39%	3.17%	6.41%
LC + G	5.7057	4.3335	2.4500	6.0696	4.9565	3.1023
	–1.34%	3.13%	8.16%	–1.26%	1.98%	10.83%
CBD + G	5.6842	4.2217	2.2783	6.0832	4.8867	2.9111
	–1.71%	0.47%	0.58%	–1.04%	0.55%	4.00%
CBD + Log	5.6800	4.1927	2.4322	6.0826	4.8542	3.0042
	–1.78%	–0.22%	7.37%	–1.05%	–0.12%	7.33%
CBD + CK	5.6865	4.2188	2.2901	6.0847	4.8844	2.9200
	–1.67%	0.40%	1.10%	–1.02%	0.50%	4.32%

TABLE C.3
Single Premium of 10-Year Pure Annuity Paying $1 Each Year—Japan

	Male			Female		
Model	70	80	90	70	80	90
Actual cost of annuity	5.9874	4.6890	2.6888	6.4584	5.5623	3.4986
LC	5.9883	4.6916	2.7974	6.4627	5.5616	3.5392
	0.02%	0.05%	4.04%	0.07%	–0.01%	1.16%
CBD	5.9441	4.7565	2.9710	6.4428	5.6021	3.7805
	–0.72%	1.44%	10.49%	–0.24%	0.72%	8.06%
LC + Log	5.9421	4.7436	3.0383	6.4376	5.5649	3.7626
	–0.76%	1.16%	13.00%	–0.32%	0.05%	7.55%
LC + CK	5.9854	4.7224	2.7465	6.4619	5.5728	3.5220
	–0.03%	0.71%	2.15%	0.05%	0.19%	0.67%
LC + G	5.9872	4.7147	2.7255	6.4624	5.5722	3.4956
	0.00%	0.55%	1.36%	0.06%	0.18%	–0.09%
CBD + G	5.9441	4.7565	2.9711	6.4428	5.6021	3.7807
	–0.72%	1.44%	10.50%	–0.24%	0.72%	8.06%
CBD + Log	5.9404	4.7483	3.0495	6.4417	5.5937	3.8292
	–0.78%	1.26%	13.41%	–0.26%	0.56%	9.45%
CBD + CK	5.9441	4.7565	2.9710	6.4428	5.6021	3.7805
	–0.72%	1.44%	10.49%	–0.24%	0.72%	8.06%

TABLE C.4
Single Premium of 10-Year Pure Annuity Paying $1 Each Year—Taiwan

	Male			Female		
Model	70	80	90	70	80	90
Actual cost of annuity	5.7768	4.4971	2.5655	5.7768	4.4971	2.5655
LC	5.8175	4.3359	2.6022	5.8175	4.3359	2.6022
	0.70%	–3.59%	1.43%	0.70%	–3.59%	1.43%
CBD	5.7461	4.4618	2.7092	5.7461	4.4618	2.7092
	–0.53%	–0.78%	5.60%	–0.53%	–0.78%	5.60%
LC + Log	5.7500	4.4972	2.8797	5.7500	4.4972	2.8797
	–0.46%	0.00%	12.25%	–0.46%	0.00%	12.25%
LC + CK	5.8175	4.3359	2.6022	5.8175	4.3359	2.6022
	0.70%	–3.59%	1.43%	0.70%	–3.59%	1.43%
LC + G	5.8115	4.4286	2.4247	5.8115	4.4286	2.4247
	0.60%	–1.52%	–5.49%	0.60%	–1.52%	–5.49%
CBD + G	5.7461	4.4618	2.7092	5.7461	4.4618	2.7092
	–0.53%	–0.78%	5.60%	–0.53%	–0.78%	5.60%
CBD + Log	5.7422	4.4588	2.8066	5.7422	4.4588	2.8066
	–0.60%	–0.85%	9.40%	–0.60%	–0.85%	9.40%
CBD + CK	5.7461	4.4618	2.7092	5.7461	4.4618	2.7092
	–0.53%	–0.78%	5.60%	–0.53%	–0.78%	5.60%