Using Graduation to Modify the Estimation of Lee-Carter Model for Small Populations

Jack C. Yue[1] Hsin-Chung Wang[2] Tzu-Yu Wang[3]

Abstract

Many mortality models, such as the Lee-Carter model (Lee and Carter 1992), have unsatisfactory estimation in the case of small populations. Increasing population size is a natural choice to stabilize the estimation, if we can find a larger reference population which has similar mortality profile as the small population. Aggregating historical data of the small populations is a fine candidate as the reference population. However, it is often not feasible in practice and we need to rely on other reference populations. In this study, we want to explore whether the graduation methods can be used if the mortality profile of small population differ from that of reference population.

In order to explore when is the appropriate occasion to use graduation methods, we create several mortality scenarios and similarity types between small and reference populations. We propose combining the graduation methods and mortality models, either graduating mortality rates first or applying mortality model first, and verify if they can improve the model fit. We use computer simulation to check if the proposed approach has better mortality estimation than the Lee-Carter model and the Li-Lee model (2005). We found that the Li-Lee model always has smaller estimation errors than the Lee-Carter model, and the proposed approach has smaller estimation errors than the Li-Lee model in most cases.

Keywords: Small area estimation, Standard mortality ratio, Graduation, Lee-Carter model, Longevity risk

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1. Introduction

Since people are living longer, life planning for the elderly has become a popular issue around the world. Among all topics, the study of the elderly's mortality rates and health receives a lot of attention. However, human life expectancy has been increasing rapidly, and in many countries (especially those with small populations and a rapid increase in longevity), data about the elderly have been limited in quantity and available period, which makes modeling mortality rates of the elderly difficult. For example, the famous Lee-Carter model (Lee and Carter 1992) does not fit well in the case of small populations (Booth et al., 2006) and the estimates of age-related parameters α_x and β_x tend to be biased. Wang et al. (2018) found that the bias is especially noticeable when the population size is 200,000 or less. For the CBD model (Cairns et al., 2006), Chen et al. (2017) found that the uncertainty of parameter estimation is related to the sample size.

The following example demonstrates the influence of small populations. We first use Taiwan's female mortality to derive the parameters of the Lee-Carter model. Suppose that the mortality rates follow the Lee-Carter model and the population structure is the same as that of the Taiwan female. We consider different population sizes, ranging from 10,000 to 5 million, and then simulate the random numbers of deaths, and then we apply them to the Lee-Carter model. To emphasize the influence of small populations, we only show the estimation results for the cases of population sizes not more than 200,000. Figure 1 shows the average biases of estimates of parameters α_x and β_x via singular value decomposition. The biases of α_x estimates are especially noticeable and always larger than 0. In contrast, the biases of β_x estimates can be positive or negative and seem to be around 0 on average, when the sample size is larger than 100,000. Note that the average biases are calculated based on 1,000 replications; the data period is 1996-2015, and the age range is 0-99 in the format of five-year age groups (20 groups).

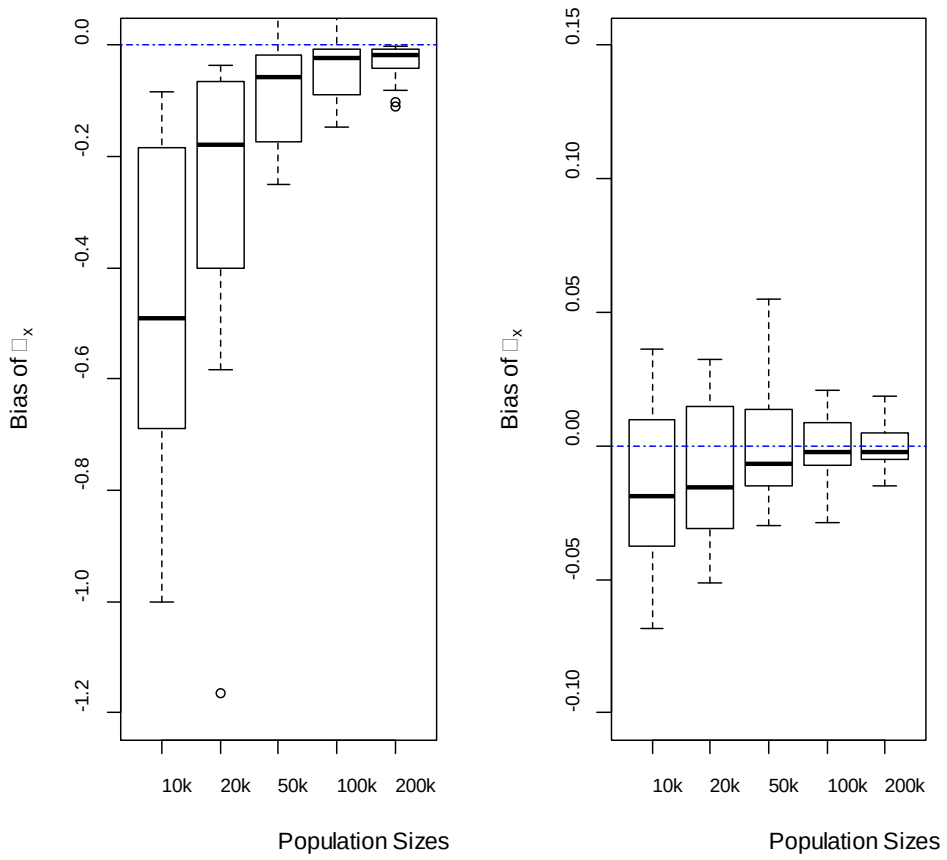

Figure 1. Bias of Parameters' Estimates of the Lee-Carter Model

The biased estimates in the case of small populations probably are the main reason why many recent studies focus on modifying mortality models for small populations. Intuitively, increasing the sample size is the most efficient way to stabilize the parameter estimation of mortality models, and including the mortality data from neighboring areas (or areas with similar mortality profiles) is a natural choice. For example, Li and Lee (2005) proposed referencing the mortality data from populations with similar mortality improvements, namely, the coherent Lee-Carter model, to reduce the estimation errors of Lee-Carter model. Ahcan et al. (2014) suggested augmenting the size of small population by including the average mortality from neighboring populations. Wang et al. (2018) proposed aggregating 10 to 20 years of historical data from the target population as the reference group. Of course, the

Bayesian approach is another possibility for increasing the sample size, such as the Bayesian modification of Lee-Carter model by Wiśniowski et al. (2015). In a sense, most studies consider increasing the sample size of small populations, but it is difficult to judge which populations have mortality profiles similar to that of the target population.

Dealing with estimating mortality rates of small populations is not new in the insurance industry, and actuaries often apply smoothing methods to reduce the fluctuations of age-specific mortality rates in constructing life tables. In fact, the graduation methods originally are designed to handle the problem of insufficient data, particularly for the elderly. Many traditional graduation methods (e.g., moving weighted averages and the Whittaker method) enlarge the sample size by including data from adjacent ages with similar mortality rates. If the fluctuations of mortality rates are reduced at each age, then the estimation of mortality model would be more stable as well. In other words, it is possible to apply the idea of graduation to stabilize the estimation of mortality models.

Thus, to deal with the estimation of mortality models in the case of small populations, we propose combining the graduation methods and mortality models (either graduate the mortality rates first or apply the mortality model first) and verify if it can improve model fit. In particular, we want to explore whether the graduation methods can be used if the mortality profile of small population differ from that of the reference population. We use computer simulation to check if the proposed approach has better mortality estimation than the Lee-Carter model (1992) and the Li-Lee model (2005). However, unlike other studies of small populations, we do not define the mortality measures for deciding which populations are similar to the small population. Instead, we create several mortality scenarios and similarity types between small and reference populations. Note that the mortality scenarios are similar to those in Wang et al. (2012) and the similarity types are defined via m_x, α_x, and β_x, which will be discussed in Section 4. We will introduce the graduation methods and mortality

models in the next two sections.

2. Methodology

The idea behind the proposed approach is similar to that of using graduation methods to adjust irregular fluctuations in observed mortality rates. However, unlike the usual graduation methods, such as the moving average, the proposed adjustment of mortality rates is based on a reference population, similar to Bayesian graduation. Basically, we propose two graduation methods: partial standard mortality ratio (SMR) and the Whittaker method. We will introduce the proposed approach in this section and evaluate its performance in the next section.

The partial SMR (Lee 2003) is a modification of SMR, which is used to smooth mortality rates of small populations via the information from a large population, referencing the value of the SMR. The SMR, which is often used in epidemiology, is defined as follows:

$$\text{SMR} = \frac{\sum_x d_x}{\sum_x e_x} = \frac{\sum_x d_x}{\sum_x P_x \times m_x^R}, \tag{2.1}$$

where d_x and e_x are the observed and expected numbers of deaths at age x for the small population, P_x is the population size of age x for the small population, and m_x^R is the central death (or mortality) rate of age x from the reference population. The SMR can be treated as a mortality index. If the SMR is larger (or smaller) than 1, then it usually indicates that the small population has a higher (or lower) overall mortality rate than the reference population.

The numbers of age-specific deaths in the small population often are not many, and the observed mortality rates fluctuate a lot and sometimes are even 0. The SMR can provide a possible guideline to fine-tune these mortality rates. For the partial SMR, the graduated mortality rates satisfy

$$v_x = u_x^* \times \exp\left(\frac{d_x \times \hat{h}^2 \times \log(d_x / e_x) + (1 - d_x / \sum d_x) \times \log(\text{SMR})}{d_x \times \hat{h}^2 + (1 - d_x / \sum d_x)} \right), \qquad (2.2)$$

or the weighted average between raw mortality rates and SMR, where $\hat{h}^2$ is the estimate of parameter h^2 for measuring the heterogeneity (in mortality rates) between the small and reference populations, and u_x^* is the mortality rate for age *x* in the reference population.

The idea behind the partial SMR is similar to a credibility-weighted estimate for calculating the future premium (Klugman et al. 2012), where the estimate is a linear combination of recent observed loss and related reference information. The Bayesian graduation methods (e.g., Kimeldorf and Jones 1967) function in a similar format, and the updated (or posterior) estimates are also a linear combination of new observations and past experience (London 1985). The key is to choose appropriate weights and the proper reference population. Of course, the reference population should have larger population size in order to have smooth values of u_x^*.

To achieve satisfactory results, Lee (2003) suggests the weight of partial SMR:

$$\hat{h}^2 = \max\left(\frac{\sum \left((d_x - e_x \times SMR)^2 - \sum d_x \right)}{SMR^2 \times \sum e_x^2} \ , \ 0 \right) \qquad (2.3)$$

The larger $\hat{h}^2$ is, the larger the difference in age-specific mortality rates (i.e., mortality heterogeneity, or larger dissimilarity in shape between the age-specific mortality curve of the small population and that of the larger population). When the number of deaths is smaller, there will be greater weight from the large population, and the graduated mortality value equals $\text{SMR} \times u_x^*$ when the number of deaths is 0. Lee mentioned that using the weight function $\hat{h}^2$ in Equation (2.3) usually has smaller mean square error (MSE) in mortality estimation. However, the derivation of $\hat{h}^2$ is through some sort of approximations, and it

cannot guarantee to have the smallest MSE.

Alternatively, we can also use the Whittaker graduation method to stabilize the mortality rates of a small population, with a modification similar to the partial SMR. First, we calculate the age-specific ratio of mortality rates from the small population to those from the reference population, or define $s_x = u_x / u_x^*$, where u_x is the observed mortality rate of age x for the small population. Next, we apply the Whittaker graduation to the mortality ratio s_x via minimizing the following objective function:

$$M = \sum_x w_x (r_x - r_x^*)^2 + h \sum_x (\Delta^z r_x^*)^2 , \qquad (2.4)$$

where r_x is the graduated mortality ratio, w_x is the weight (or exposure) of age x, h is a smoothing parameter, and Δ is the difference operator, or $\Delta f(x) = f(x + 1) – f(x)$. Finally, the graduated mortality rates of small population are $s_x \times u_x^*$. The choice of parameter h is the key, as well as the choice of reference population, in applying the Whittaker ratio (namely) graduation.

Selecting the reference population is critical in applying the proposed graduation methods. This is also the case for applying the coherent Lee-Carter model, and choosing the appropriate group of coherent populations is important. In practice, selecting the populations with similar mortality profiles is not easy, and a natural choice is the whole nation (or nearby areas) if the small population is a subset of the nation. But the mortality differences within a country can be huge, even for neighboring cities. For example, in Taiwan, the largest difference in life expectancy between counties is more than 10 years (the Taipei City versus Tai-tung County in the 2014 Taiwan Abridged Life Tables). It would be questionable to use the population of Taipei City as the reference group for Tai-tung County. In the next section, we will use computer simulation to evaluate the proposed approach, with emphasis on the similarity between the small and reference populations.

3. Graduating Mortality Rates via the Reference Population

As mentioned previously, choosing the appropriate reference population is important. However, instead of searching for the perfect reference population, we want to use the similarity level between the small and reference populations to judge whether we should adjust the mortality rates of the small population via the reference population. In this section, we first evaluate the performance of graduation methods using various similarity levels. In the next section, we will use the graduation to integrating the parameter estimation of the Lee-Carter model.

Suppose that there are seven scenarios for the mortality ratio s_x between the small and reference populations, as shown in Figure 2. Various scenarios are designed to evaluate the effect of different graduation methods. The three scenarios in the left panel indicate that the mortality rates of the small and reference populations are similar, and we expect that the partial SMR would be a good choice for graduation. In contrast, the other four scenarios in the right panel assume that the mortality rates of the small and reference populations are different. For these four cases, the partial SMR might not be a good choice.

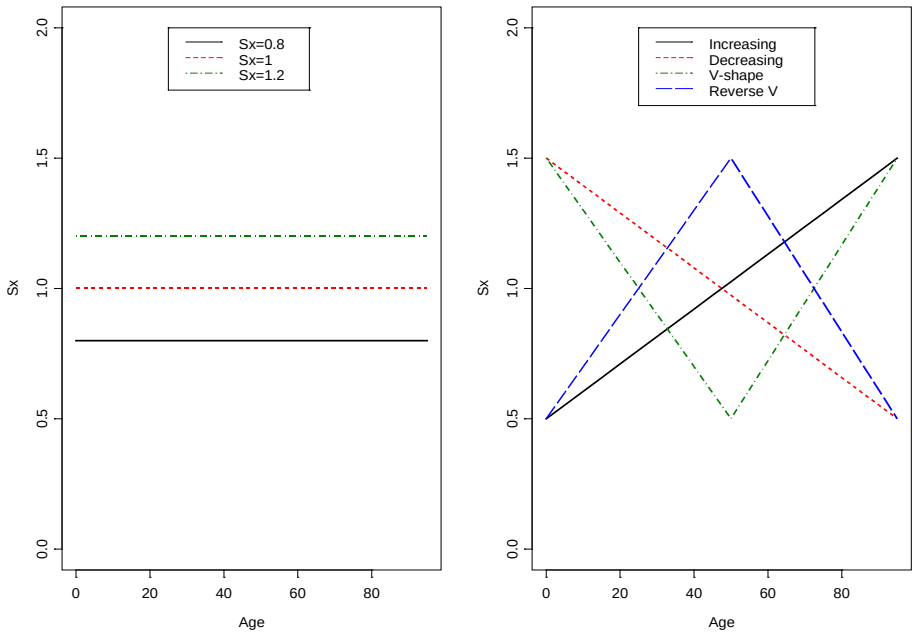

Figure 2. Seven Mortality Ratio Scenarios

We use a computer simulation to evaluate the performance of partial SMR and Whittaker graduation. First, we use the mortality data of 1996-2015 Taiwanese female, with age range 0-99 in the format of five-year age groups of 0-4, 5-9, 10-14, …, 95-99 (20 groups), to obtain parameters (α_x, β_x, and k_t) of Lee-Carter model. Second, we adopt the age structure of 1996-2015 Taiwanese female population as the underlying population and the population sizes are 10,000, 20,000,…, 2 million, 5 million. Third, we simulate the numbers of deaths from Poisson distribution for both the small and reference populations under seven mortality scenarios. Then, we calculated raw mortality rates and apply them into the graduation methods, the Lee-Carter model, and the Li-Lee models. The preceding simulation process is repeated for 1,000 times.

There are quite a lot of possible treatment combinations and we should use two examples as a demonstration. Suppose the size of the small population is 100,000 or 200,000, and the size of reference population is 2 million or 5 million. Also, the comparison criterion is based on the mean absolute percentage error (MAPE):

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|Y_i - \hat{Y}_i\right|}{Y_i} \times 100\%, \tag{3.1}$$

where Y_i and $\hat{Y}_i$ are the observed and predicted values for observation *i*, *i* = 1, 2, ..., *n*. According to Lewis (1982), a prediction with MAPE less than 10 percent is treated as highly accurate, and a MAPE greater than 50 percent is considered inaccurate.

Since the simulation results are similar for the cases where the reference population is larger than 2 million, we will only show the cases of 2 million. Tables 1 and 2 are the simulation results of cases where the small population is 100,000 and 200,000, for 1,000 simulation replications. Other than raw data and two proposed graduation methods, we also

consider the case of Whittaker graduation to the observed mortality rates as a control group. For the Whittaker ratio and Whittaker graduation methods, the parameter w_x is the exposure of age x, and the parameter h is average exposure of all ages.

As expected, the graduation methods generally have smaller MAPEs than those without graduation (except for increasing and reverse V shape scenarios). For the first three scenarios, in which the mortality rates of small and reference populations have the same proportion for all ages, the SMR can provide a very good approximate estimate of this proportion. Thus, Table1 and Table 2 shows the MAPEs of the partial SMR are much smaller than other methods. Heuristically speaking, taking the results in Table 1 as a demonstration, it is like treating the reference population as the small population when we apply the partial SMR, so the MAPEs of the raw data are about 2 times of those for the partial SMR.

Table 1. MAPEs of Graduation Methods (100,000 vs. 2 Million)

	s_x = **0.8**	s_x = **1**	s_x = **1.2**	**Increase**	**Decrease**	**V**	**Rev-V**
Raw	28.95%	26.80%	25.08%	28.96%	26.73%	27.28%	28.24%
Whittaker	26.73%	24.77%	23.50%	28.39%	23.60%	27.15%	25.67%
Whittaker ratio	15.73%	14.95%	14.30%	19.46%	18.36%	15.54%	17.44%
Partial SMR	12.80%	12.43%	12.08%	47.65%	20.12%	22.15%	25.33%

Note: The cells with gray background are those with the smallest MAPEs.

Table 2. MAPEs of Graduation Methods (200,000 vs. 2 Million)

	s_x = **0.8**	s_x = **1**	s_x = **1.2**	**Increase**	**Decrease**	**V**	**Rev-V**
Raw	21.42%	19.69%	18.34%	21.54%	19.52%	19.99%	20.92%
Whittaker	21.79%	20.54%	19.59%	23.03%	19.64%	23.66%	20.27%
Whittaker ratio	12.89%	12.15%	11.58%	15.01%	14.83%	12.50%	14.04%
Partial SMR	11.58%	11.20%	10.87%	43.10%	17.40%	16.99%	21.77%

Note: The cells with gray background are those with the smallest MAPEs.

For the other four mortality scenarios, where the mortality rates of small and reference

populations are not very similar, the MAPEs of the Whittaker ratio generally are the smallest. It seems that the Whittaker ratio is more robust than the partial SMR and the graduation results are not influenced much by different mortality scenarios. This probably can explain why the Whittaker method is still a popular choice of graduation methods.

Of course, we can conduct exploratory data analysis (EDA) to evaluate if the mortality rates of small and reference populations are similar. For example, the age-specific mortality ratios in Figure 2 are one of the EDA tools we can use. We suggest using the partial SMR if they look like the first three scenarios, but we are skeptical of using the partial SMR for the last four scenarios. In fact, we experimented with using different values of mortality ratios for the last four scenarios, such as changing the ratios of the increasing scenario from 0.5~1.5 to $(1 - a)$~$(1 + a)$ for $0 < a < 1$. We found that the MAPEs of the partial SMR are smaller than those of Whittaker ratio if $a \leq 0.4$. In other words, if the small and reference populations are not very different, then the partial SMR is preferred. We should continue exploring whether we can modify the stochastic mortality models via graduation methods in the next section.

4. Modification of the Lee-Carter Model

In this section, we continue the discussion of applying the graduation methods to modify the Lee-Carter model. We first use the proposed approach to smooth the mortality rates and then apply the graduated mortality rates to fit the Lee-Carter model. We assume that the age-specific mortality rates of small and reference populations satisfy the Lee-Carter model. In addition, three different mortality settings are applied to regulate the relationship between the small and reference populations, via mortality ratio, α_x, and β_x. The purpose behind these settings is to explore the influence of similarity in the mortality rates and their trend between the small and reference populations. For example, the mortality rates look similar now but their improvement rates may not. This would cause mortality discrepancy and possible bias in the parameter estimation.

As for the proposed approach, we can either graduate raw mortality rates first and then apply the mortality model, or apply the mortality model first and then graduate model-fitted mortality rates. The graduation methods considered are the partial SMR and Whittaker ratio, and the mortality models are the Lee-Carter and Li-Lee models. In addition to the Lee-Carter and Li-Lee models, we also compare the proposed approach to the method of aggregate historical data (20 years) of small population (Wang et al. 2018). We use computer simulation to evaluate whether the proposed approach can improve the model fit. We only use the case where the size of small population is 100,000 and that of reference population is 2 million as a demonstration, since there are quite a few treatment combinations (order of mortality graduation, graduation methods, and mortality models). For example, for the case of partial SMR + Li-Lee model, we first simulate numbers of deaths for small and reference populations. Then, we use the partial SMR method to smooth the mortality rates of small population, using the information of mortality rates from the reference population. Finally, we apply the Li-Lee Carter model to the graduated mortality rates of small population and observed mortality rates of reference population.

Again, we use the age structure of Taiwanese female population as the underlying population. The model comparison is based on the MAPE as well, based on 1,000 simulation runs. Because there are many treatment combinations, we only show the results of approaches for graduating raw mortality rates first and then applying the mortality model. The results of approaches for applying the mortality model first and then graduating model-fitted mortality rates are in the Appendix. We first consider the case of mortality ratio $s_x = m_x^s / m_x^R$, where m_x^s and m_x^R are the central mortality rates of small and reference populations. The MAPEs of the proposed methods and the reference group are shown in Table 3 and Table A-1 in the Appendix. Note that the last rows of Table 3 have two different reference populations. The reason is that there are only one set (or “single year”) of mortality

data for the reference population after data aggregation, and thus the Li-Lee model cannot be used. As an alternative, we need to reply on other reference population after applying graduation methods. Similar rule applies to the last two rows of Table A-2. As expected, the MAPEs of Lee-Carter and Li-Lee models are obviously smaller than those of raw observations, since the mortality rates satisfy the Lee-Carter model. In addition, the MAPEs of Li-Lee model are always smaller than those of Lee-Carter model. It is interesting to note that the Li-Lee model is a fine modification to the Lee-

Table 3. MAPEs of Graduation and Lee-Carter Model (Mortality Ratio: $s_x = m_x^s / m_x^R$)

	Reference	**0.8**	**1**	**1.2**	**Incr.**	**Decr.**	**V**	**Rev-V**
Raw	---	28.95	26.80	25.08	28.96	26.73	27.28	28.24
Lee-Carter	---	16.48	14.62	13.61	18.45	14.91	14.68	17.95
Li-Lee	2 mil.	13.94	12.74	12.05	15.77	11.99	13.24	14.22
Partial SMR + Lee-Carter	2 mil.	4.92	4.69	4.46	39.49	23.86	20.59	22.62
Whittaker Ratio + Lee-Carter	2 mil.	9.08	8.72	8.59	13.67	16.56	10.73	11.33
Partial SMR + Li-Lee	2 mil.	5.06	4.79	4.59	39.50	23.78	20.57	22.72
Whittaker Ratio +Li-Lee	2 mil.	8.65	8.34	8.16	12.11	16.48	10.74	10.33
Partial SMR + Lee-Carter	Aggregate	10.47	9.79	9.22	10.51	9.57	9.69	10.27
Whittaker Ratio + Lee-Carter	Aggregate	8.97	8.62	8.51	8.84	8.99	9.19	8.58
Partial SMR + Li-Lee	Aggregate / 2 mil.	9.77	9.20	8.74	10.05	8.81	9.13	9.68
Whittaker Ratio +Li-Lee	Aggregate / 2 mil.	10.91	10.07	9.53	11.34	9.93	10.32	10.77

Note: Cells with gray background are those with the smaller MAPE than the Li-Lee model.

Carter model, even though the reference population has quite different mortality rates. A possible explanation is that the relationship between the mortality rates of small and reference population are fixed in this setting, i.e., two populations have same β_x and κ_t, and it is like fitting the Li-Lee model using both the small and reference populations.

The MAPEs of graduation methods vary quite a lot. For the first three mortality scenarios, using the partial SMR to graduate first and then apply the Lee-Carter model has the smallest MAPEs, but the MAPEs for the last four scenarios are almost the largest. In contrast, applying the mortality model (Lee-Carter or Li-Lee model) first and using Whittaker ratio graduation outperforms the Lee-Carter model. The treatment combination Whittaker ratio + Li-Lee model has the smallest MAPEs for the last four mortality scenarios, with noticeable improvements (at least 30%) over the Lee-Carter and Li-Lee models in all mortality scenarios. It seems that the graduation methods can improve the mortality estimates of small populations if they are chosen properly.

Table 4. MAPEs of Graduation and Lee-Carter Model (Different $\alpha_x : s_x = \alpha_x^s / \alpha_x^R$)

	Reference	**0.8**	**1**	**1.2**	**Incr.**	**Decr.**	**V**	**Rev-V**
Raw	---	66.01	26.76	15.33	317.47	17.66	27.85	173.52
Lee-Carter	---	54.05	14.63	8.26	311.48	10.25	14.56	168.31
Li-Lee	2 mil.	50.24	12.77	7.51	304.14	8.20	13.06	161.25
Partial SMR + Lee-Carter	2 mil.	57.07	9.44	19.72	1198.31	8.37	96.21	411.76
Whittaker Ratio + Lee-Carter	2 mil.	40.21	9.03	20.80	297.34	36.62	31.90	178.11
Partial SMR + Li-Lee	2 mil.	57.04	9.28	19.64	1198.84	9.35	96.61	411.51
Whittaker Ratio +Li-Lee	2 mil.	39.25	8.60	20.44	292.73	36.83	30.85	175.71
Partial SMR + Lee-Carter	Aggregate	16.66	9.81	6.82	24.80	7.16	9.79	17.56
Whittaker Ratio + Lee-Carter	Aggregate	20.64	10.69	6.44	27.33	6.30	10.54	20.01
Partial SMR + Li-Lee	Aggregate / 2 mil.	15.84	9.21	6.69	24.61	7.00	9.55	17.49
Whittaker Ratio +Li-Lee	Aggregate / 2 mil.	19.60	10.11	6.39	27.39	7.31	10.91	19.89

Note: Cells with gray background are those with the smaller MAPE than the Li-Lee model

Following the same concept, we also set up seven mortality scenarios for the parameter α_x to describe the relationship between the small and reference populations. Let

$s_x = \alpha_x^s / \alpha_x^R$, where α_x^s and α_x^R are the intercept parameters of the Lee-Carter model for the small and reference populations. The MAPEs of various model estimations are shown in Table 4 and Table A-2 in the Appendix, and the results for the setting of different α_x (same β_x and κ_t) are very different, comparing to those in Table 3. We can see that the MAPE values of all methods (except the aggregate methods) are especially larger for the scenarios of Increasing and Rev-V. It seems that the discrepancy in the intercept α_x causes noticeable influence for the parameter estimation of the small population. This suggests that, if the mortality profiles of small and reference populations differ a lot, we should apply the mortality models and graduation methods with care.

The simulation for the case of different β_x (same α_x and κ_t) is conducted similarly and the results are shown in Table 5 and Table A-3 in the Appendix. Let $s_x = \beta_x^s / \beta_x^R$, where β_x^s and β_x^R are the age-related slope parameters of the Lee-Carter model for the small and reference populations. Again, the Li-Lee model always has smaller MAPEs than the Lee-Carter model in all seven β_x scenarios. It seems that even if the small and reference populations have quite different β_x, using the idea of coherent group to increase the population size still can reduce the estimation error of mortality rates for the small populations. It indicates that increasing the population size is a feasible approach, even though the populations included do not have a mortality profile identical to that of the small population.

Unlike the case of different α_x, the proposed approaches have fine performance in all scenarios. The partial SMR + Lee-Carter model has the smallest MAPEs, and it outperforms the Lee-Carter and Li-Lee models in all cases, significantly reducing the estimation errors. Other methods of applying the graduation method first also have smaller MAPEs than those of the Lee-Carter and Li-Lee models, but the error reduction is not as significant. This is very

similar to those in Table 3 and very different than those in Table 4. Intuitively, like in the regression analysis, we think that the slope parameter β_x should play a more important role than the intercept parameter α_x, but the MAPEs of computer simulation show different information.

Table 5. MAPEs of Graduation and Lee-Carter Model (Different β_x : $s_x = \beta_x^s / \beta_x^R$)

	Reference	**0.8**	**1**	**1.2**	**Incr.**	**Decr.**	**V**	**Rev-V**
Raw	---	26.83	26.72	26.81	26.76	26.78	26.80	26.79
Lee-Carter	---	14.93	14.69	14.12	15.28	13.95	14.20	14.67
Li-Lee	2 mil.	12.99	12.75	13.30	13.32	13.34	13.38	13.30
Partial SMR + Lee-Carter	2 mil.	4.69	4.68	4.85	6.48	6.25	5.65	5.81
Whittaker Ratio + Lee-Carter	2 mil.	8.99	8.88	8.75	9.22	8.80	8.74	8.90
Partial SMR + Li-Lee	2 mil./ 2 mil.	5.36	4.79	5.32	6.50	6.36	5.96	6.12
Whittaker Ratio +Li-Lee	2 mil./ 2 mil.	8.72	8.41	8.90	9.37	9.03	8.98	9.17
Partial SMR + Lee-Carter	Aggregate	9.03	9.76	10.64	9.42	12.10	11.36	9.52
Whittaker Ratio + Lee-Carter	Aggregate	10.86	10.78	10.65	10.51	10.88	10.83	10.41
Partial SMR + Li-Lee	Aggregate / 2 mil.	8.27	9.17	10.21	9.14	11.41	10.69	9.08
Whittaker Ratio +Li-Lee	Aggregate / 2 mil.	10.10	10.13	10.73	10.52	10.83	10.79	10.42

Note: Cells with gray background are those with the smaller MAPE than the Li-Lee model.

In summary, we found that the mortality graduation can improve the mortality estimation of the Lee-Carter model (and Li-Lee model as well), if proper graduation methods are selected. For example, the method Whittaker ratio + Li-Lee always has smaller MAPEs than the Lee-Carter and Li-Lee models in all simulation cases. However, the selection of treatment combination (i.e., graduation vs. mortality model) depends on the characteristics of mortality rates. We suggest conducting exploratory data analysis for the mortality rates, and the information, such as mortality ratios, can provide a useful guideline to choose the appropriate graduation methods.

5. Conclusion and Discussion

Living longer is a common phenomenon of human beings in the 21st century, and the study of mortality rates is a popular research topic in many fields, such as demography and actuarial science. The mortality models are a common tool for modeling the mortality rates, but the model estimation tends to be distorted by small sample size. In addition to larger variance, parameters' estimates for the small populations often are biased. Quite a lot of modifications have been proposed to deal with the case of a small population. Three examples are the coherent Lee-Carter model by Li and Lee (2005), the Bayesian approach by Cairns et al. (2011), and the SAINT model by Jarner and Kryger (2011). Most of these modifications use mortality information from another population(s) as a reference to improve the model fitting.

Including another population as a reference is like increasing the sample size, and this probably is the most intuitive and effective way to deal with the model estimation for small populations. The idea of increasing sample size has been used by actuaries to construct life tables as well, and many graduation methods can be treated as increasing sample size from those with a similar mortality profile. In this study, we adapt the idea of graduation and propose a modification of the Lee-Carter model, also with information from a reference population. Two types of graduation methods are used in this study: the partial SMR (Lee 2003) and Whittaker ratio.

We consider three settings of relationship between small and reference populations: $s_x = m_x^s / m_x^R$, $s_x = \alpha_x^s / \alpha_x^R$, and $s_x = \beta_x^s / \beta_x^R$, and use computer simulation to evaluate the proposed approach. In general, the partial SMR modification has smaller estimation errors (with respect to MAPE) than the Lee-Carter and Li-Lee models, if the small and reference populations have similar mortality profiles. When the mortality rates of small and reference

populations are not similar, the Whittaker ratio is a possible alternative choice of graduation methods. We think the graduation methods are a feasible approach for dealing with small populations and can effectively reduce the estimation errors of the Lee-Carter model.

We should continue exploring the graduation methods and use them to modify the mortality models. However, we only consider various settings of age-related parameters α_x and β_x for the Lee-Carter model and do not consider the time-related parameter κ_t. There would be problems if the small and reference populations have different functional forms of parameter κ_t (e.g., quadratic for the small population and linear for the reference population). Of course, the interactive effects might also exist if two or three parameters (α_x, β_x and κ_t) are different, and this can distort or even ruin the effect of graduation.

Also, there can be more than one reference population, and of course, it is impossible that these populations are perfectly homogeneous in terms of mortality rates. It is more realistic to expect that some populations and the small population have similar mortality rates at younger ages, while other populations and the small population are similar at older ages. Then the concept of variable selection can be applied. We may develop similarity measures and use them to judge whether a reference population should be included. Further, it would be even better (but more difficult) if the selection of appropriate reference populations is age dependent.

Modifying the graduation methods for mortality models (other than the Lee-Carter mode) is also a possible direction for future study. If the parameters of mortality models are additive, such as the age-period-cohort model, we can use the graduation methods to adjust the parameter estimation one parameter at a time. However, if the parameters are not additive, the situation is expected to be more complicated. For example, the cohort modification to the Lee-Carter model by Renshaw and Haberman (2006) contains one component of age with time and one component of age with cohort. These two components are not linearly dependent and can cause problems of adjusting the age parameters associated with time and

cohort.

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Appendix: Errors of First Applying Mortality Model and then Graduation

Table A-1. MAPEs of Lee-Carter Model & Graduation (Mortality Ratio: $s_x = m_x^s / m_x^R$)

	Reference	**0.8**	**1**	**1.2**	**Incr.**	**Decr.**	**V**	**Rev-V**
Raw	---	28.95	26.80	25.08	28.96	26.73	27.28	28.24
Lee-Carter	---	16.48	14.62	13.61	18.45	14.91	14.68	17.95
Li-Lee	2 mil.	13.94	12.74	12.05	15.77	11.99	13.24	14.22
Lee-Carter + Partial SMR	2 mil.	15.07	14.46	13.96	33.58	32.46	22.70	25.55
Lee-Carter + Whittaker Ratio	2 mil.	13.49	12.42	11.90	16.16	15.22	12.77	15.25
Li-Lee + Partial SMR	2 mil.	13.07	12.58	12.12	31.29	31.22	21.02	23.92
Li-Lee + Whittaker Ratio	2 mil.	10.29	9.44	9.05	12.86	11.80	10.21	11.22
Lee-Carter + Partial SMR	Aggregate	13.50	12.59	12.03	13.91	12.99	12.38	14.26
Lee-Carter + Whittaker Ratio	Aggregate	14.36	12.73	11.95	15.85	12.98	12.76	15.41
Li-Lee + Partial SMR	2 mil./ Aggregate	10.38	9.78	9.44	10.89	9.47	9.96	10.24
Li-Lee + Whittaker Ratio	2 mil./ Aggregate	10.95	9.78	9.15	12.40	9.18	10.07	11.07

Note: Cells with gray background are those with smaller MAPE than the Li-Lee model.

Table A-2. MAPEs of Lee-Carter Model & Graduation (Different $\alpha_x : s_x = \alpha_x^s / \alpha_x^R$)

	Reference	**0.8**	**1**	**1.2**	**Incr.**	**Decr.**	**V**	**Rev-V**
Raw	---	66.01	26.76	15.33	317.47	17.66	27.85	173.52
Lee-Carter	---	54.05	14.63	8.26	311.48	10.25	14.56	168.31
Li-Lee	2 mil.	50.24	12.77	7.51	304.14	8.20	13.06	161.25
Lee-Carter + Partial SMR	2 mil.	62.10	14.45	21.39	1224.10	10.35	87.38	464.70
Lee-Carter + Whittaker Ratio	2 mil.	43.83	12.33	20.09	301.41	38.42	31.60	180.15
Li-Lee + Partial SMR	2 mil.	60.16	12.57	20.57	1204.93	9.17	85.38	452.22
Li-Lee + Whittaker Ratio	2 mil.	42.27	9.44	18.96	293.59	37.39	30.78	176.28
Lee-Carter + Partial SMR	Aggregate	19.25	12.59	8.55	28.43	10.48	13.10	22.10
Lee-Carter + Whittaker Ratio	Aggregate	24.21	12.77	7.49	31.35	9.03	12.93	24.21
Li-Lee + Partial SMR	2 mil./ Aggregate	15.36	9.15	6.84	24.69	7.80	9.93	17.67
Li-Lee + Whittaker Ratio	2 mil./ Aggregate	21.46	9.86	6.00	27.84	6.19	10.39	20.18

Note: Cells with gray background are those with the smaller MAPE than the Li-Lee model.

Table A-3. MAPEs of Lee-Carter Model &Graduation (Different β_x : $s_x = \beta_x^s / \beta_x^R$)

	Reference	**0.8**	**1**	**1.2**	**Incr.**	**Decr.**	**V**	**Rev-V**
Raw	---	26.83	26.72	26.81	26.76	26.78	26.80	26.79
Lee-Carter	---	14.93	14.69	14.12	15.28	13.95	14.20	14.67
Li-Lee	2 mil.	12.99	12.75	13.30	13.32	13.34	13.38	13.30
Lee-Carter + Partial SMR	2 mil.	10.66	10.97	10.81	12.07	11.04	11.03	11.47
Lee-Carter + Whittaker Ratio	2 mil.	12.35	12.50	12.14	13.06	11.95	12.06	12.57
Li-Lee + Partial SMR	2 mil.	7.61	7.41	8.12	8.44	8.48	8.36	8.27
Li-Lee + Whittaker Ratio	2 mil.	9.79	9.47	10.03	10.30	10.22	10.12	10.14
Lee-Carter + Partial SMR	Aggregate	11.67	12.61	13.15	12.57	13.76	13.47	12.30
Lee-Carter + Whittaker Ratio	Aggregate	12.67	12.85	12.52	13.28	12.26	12.48	12.78
Li-Lee + Partial SMR	2 mil./ Aggregate	8.39	9.11	10.58	8.87	11.13	10.73	9.02
Li-Lee + Whittaker Ratio	2 mil./ Aggregate	10.07	9.84	10.43	10.58	10.51	10.49	10.46

Note: Cells with gray background are those with the smaller MAPE than the Li-Lee model.