統計計算與模擬

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第二單元：隨機變數模擬

http://csyue.nccu.edu.tw

以下分配的亂數如何產生？

The 10 examples of *Discrete Random Variables* are the following;

1. The number of outcomes of tossing a fair coin.
2. The number of students inside the classroom.
3. The number of honors during the school year.
4. The number of covid cases on a daily basis.
5. The number of patients in a ward.
6. The number of vaccine dosages.
7. The number of eggs sold in a day.
8. The number of recoveries from Novel Corona Virus 19 in a week.
9. The number of equations used to solve a problem.
10. The number of items during the examination.

The 10 examples of *Continuous Random Variables* are the following;

1. The distance from your school to your home.
2. The minimum salary of an employee.
3. The height requirement to become a flight attendant.
4. The minimum weight before obesity.
5. The average grades you during a semester.
6. The amount you invested for the future.
7. The temperature during the wet season.
8. The minimum temperature to store the vaccines.
9. The amount of water that a box can contain.
10. The amount of air pressure in a tank.

https://brainly.ph/question/2493362

常見的隨機變數產生方式 (Random Numbers from certain Distribution)

均勻分配以外的隨機變數通常藉由下列方法，透過均勻亂數產生。

- Inverse Transform Method
- Composition Method
- Rejection (and Acceptance) Method
- Alias Method
- Table Method

註：二維以上變數的模擬也類似，但需要矩陣的輔助，將在下一單元介紹。

Normal Distribution

- 與均勻分配類似，常態分配是最常用到的分配，常見的常態分配亂數產生法：

(Random numbers from normal distribution is one of the popular choices for simulation.)

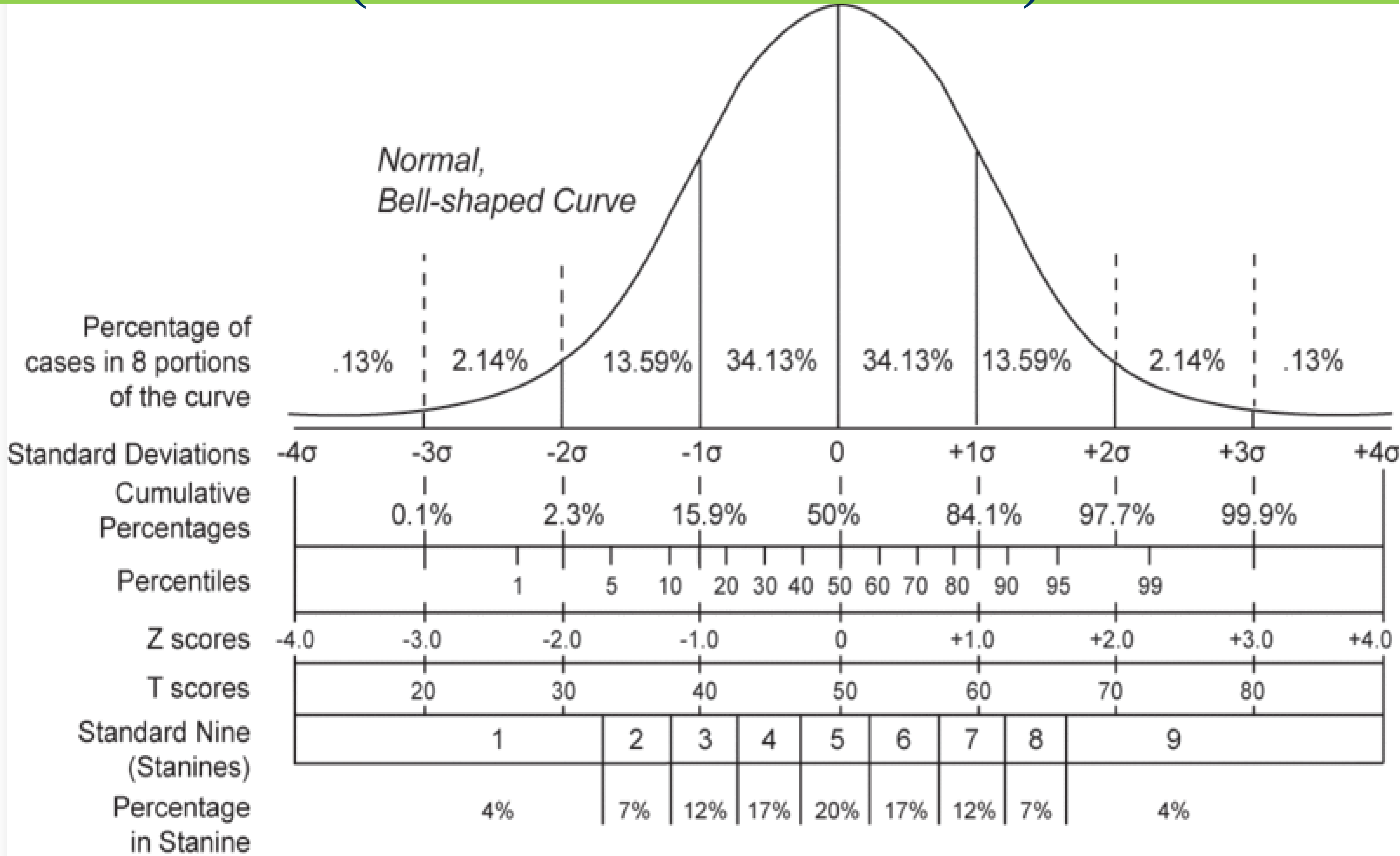
→ $\sum_{i=1}^{12} U_i - 6$ (僅為近似，only approximation)

→ Box-Muller

→ Polar

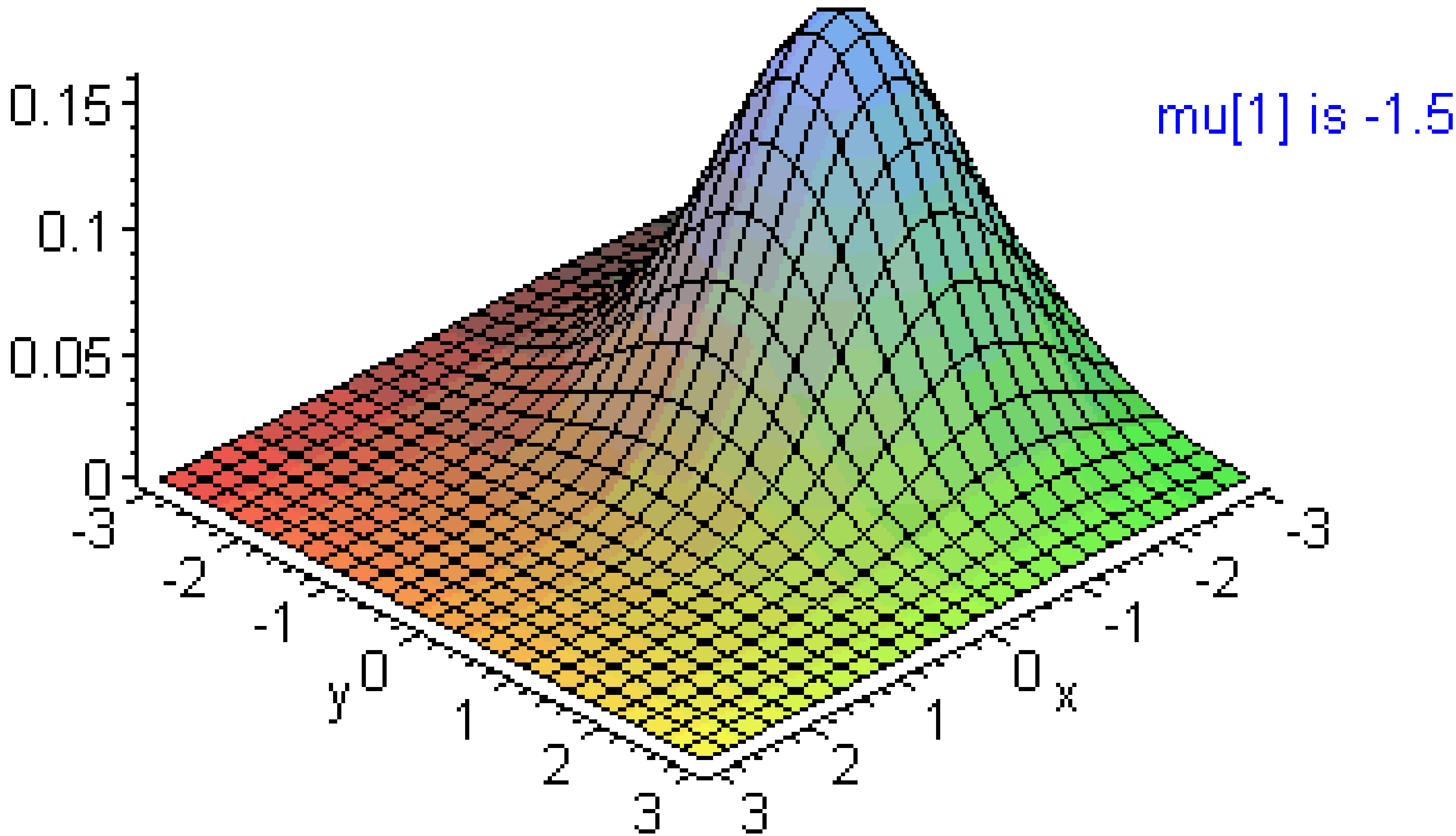
→ Ratio-of-uniforms

常態分配的重要特性
(About normal distribution)



https://upload.wikimedia.org/wikipedia/commons/b/bb/Normal_distribution_and_scales.gif

二維與多維常態分配
(Bivariate and Multivariate Normal Distribution)



https://www.statisticshowto.com/wp-content/uploads/2015/09/Bivariate41.gif

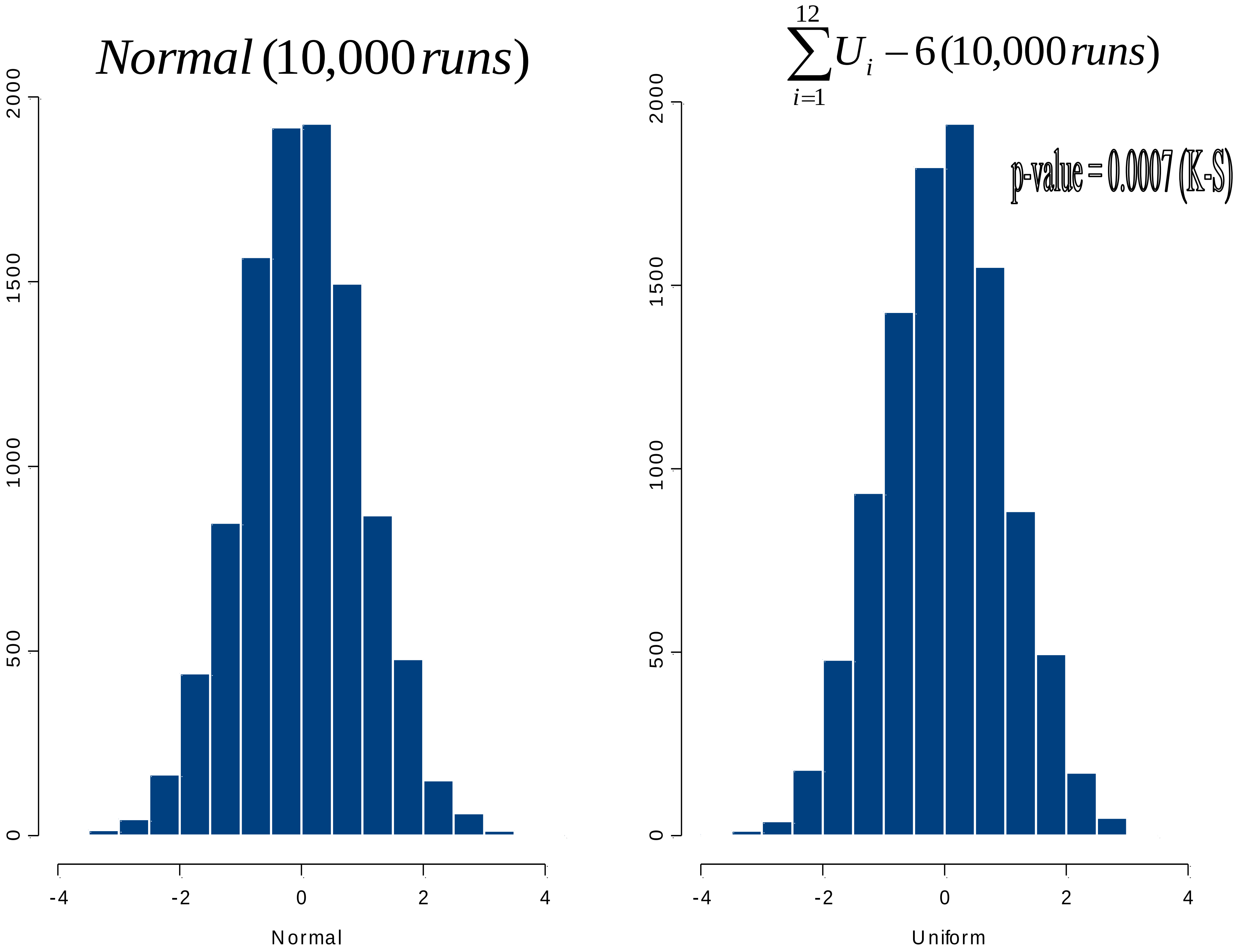
- $Y=\sum_{i=1}^{12}U_i-6$ (近似法)

→因為Y 的期望值與變異數等於

$$E(Y)=E(\sum_{i=1}^{12}U_i)-6=6-6=0$$

$$Var(Y)=12\cdot Var(U_i)=12\cdot\frac{1}{12}=1$$

與標準常態分配相同。



$Normal\,(10{,}000\, runs)$
2000
1500
1000
500
0
-4
-2
0
2
4
Normal
$\sum_{i=1}^{12} U_i - 6\,(10{,}000\, runs)$
p-value = 0.0007 (K-S)
2000
1500
1000
500
0
-4
-2
0
2
4
Uniform

- Box and Muller (1958)

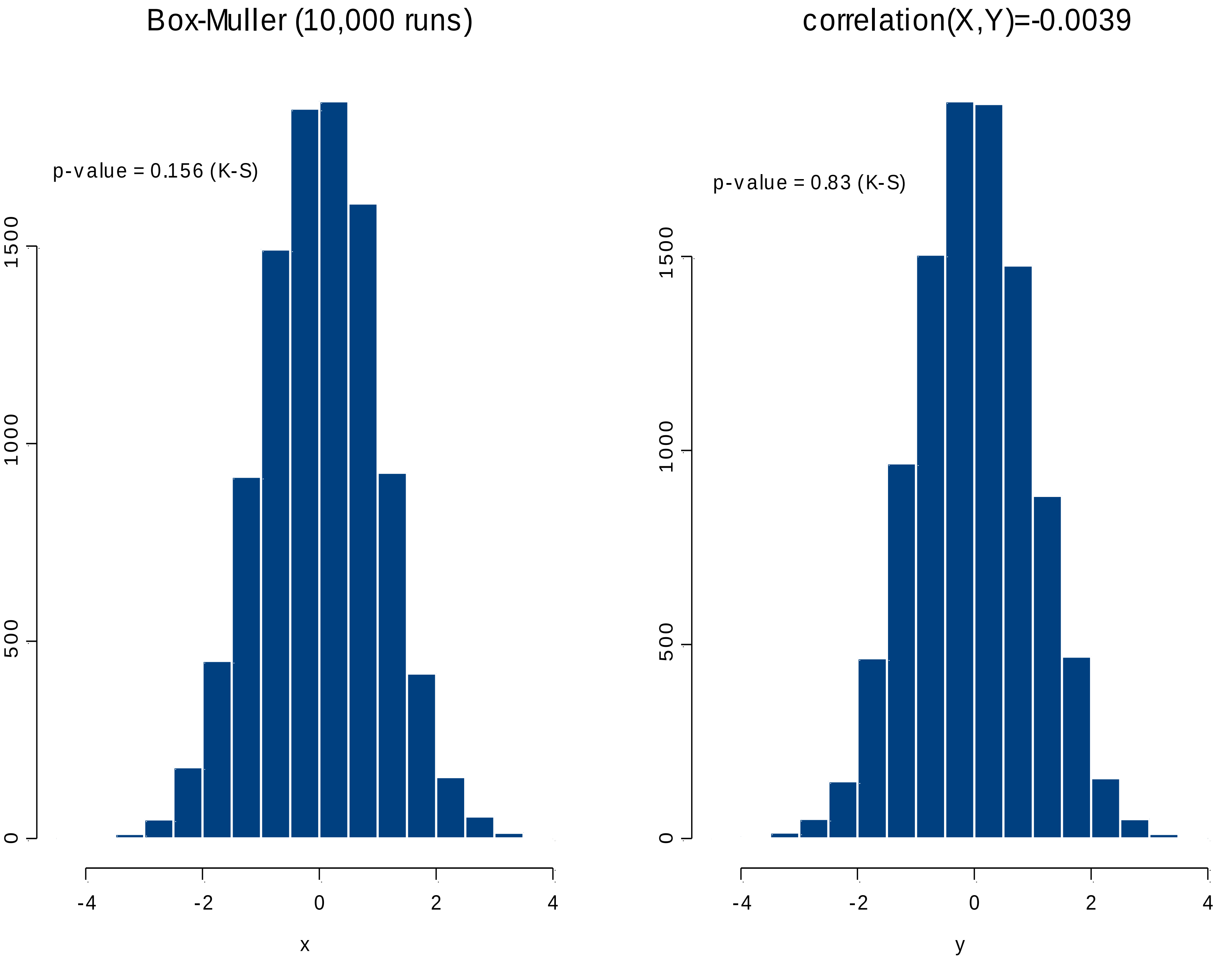
→The best known "exact" method for the normal distribution.

- Algorithm

1. Generate $U_1, U_2 \sim U(0,1)$
2. Let $\theta = 2\pi U_1$

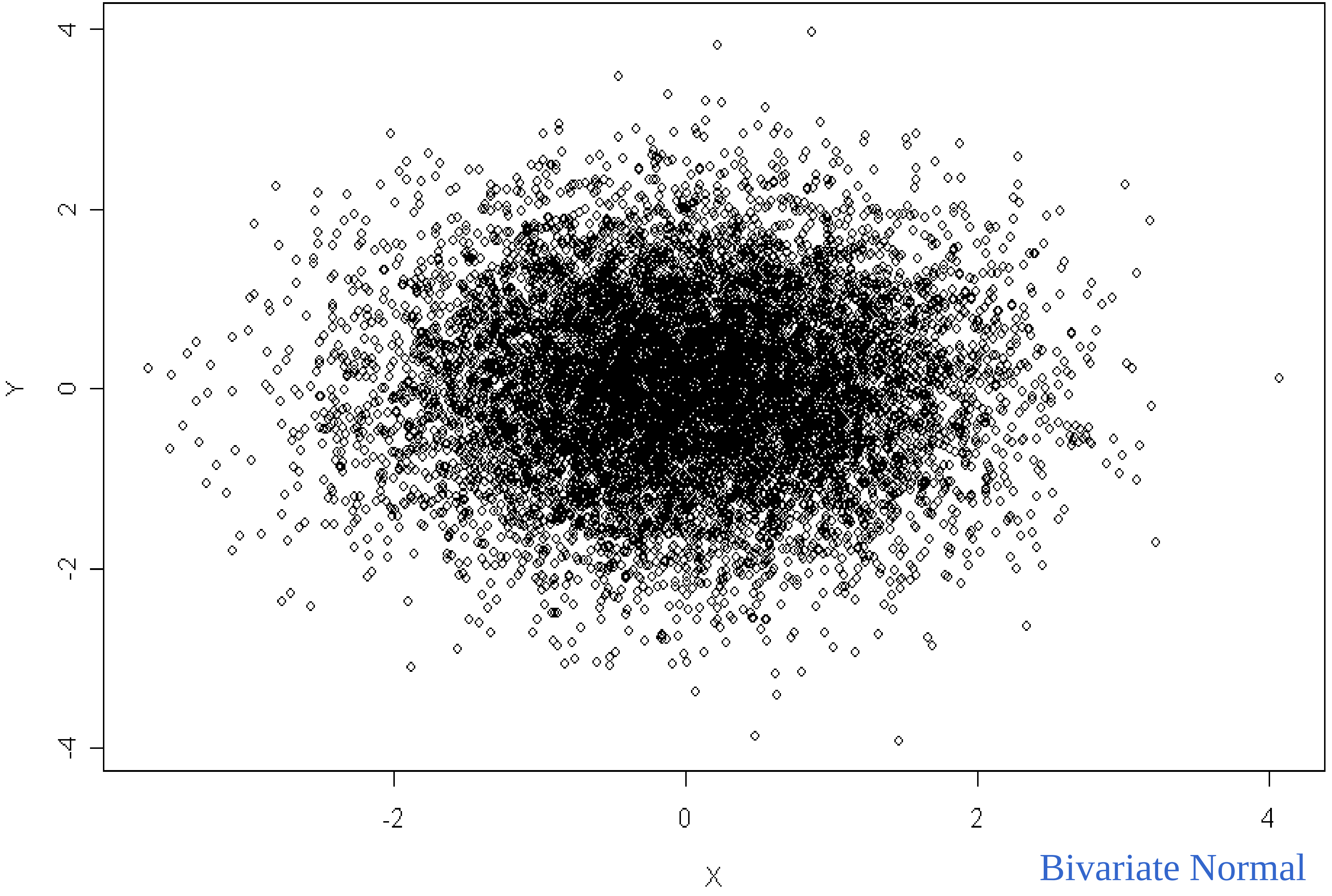
$$E = -\log U_2 \;\&\; R = \sqrt{2E}$$

3. Then $X = R\cos\theta$ and $Y = R\sin\theta$ are independent standard normal variables.



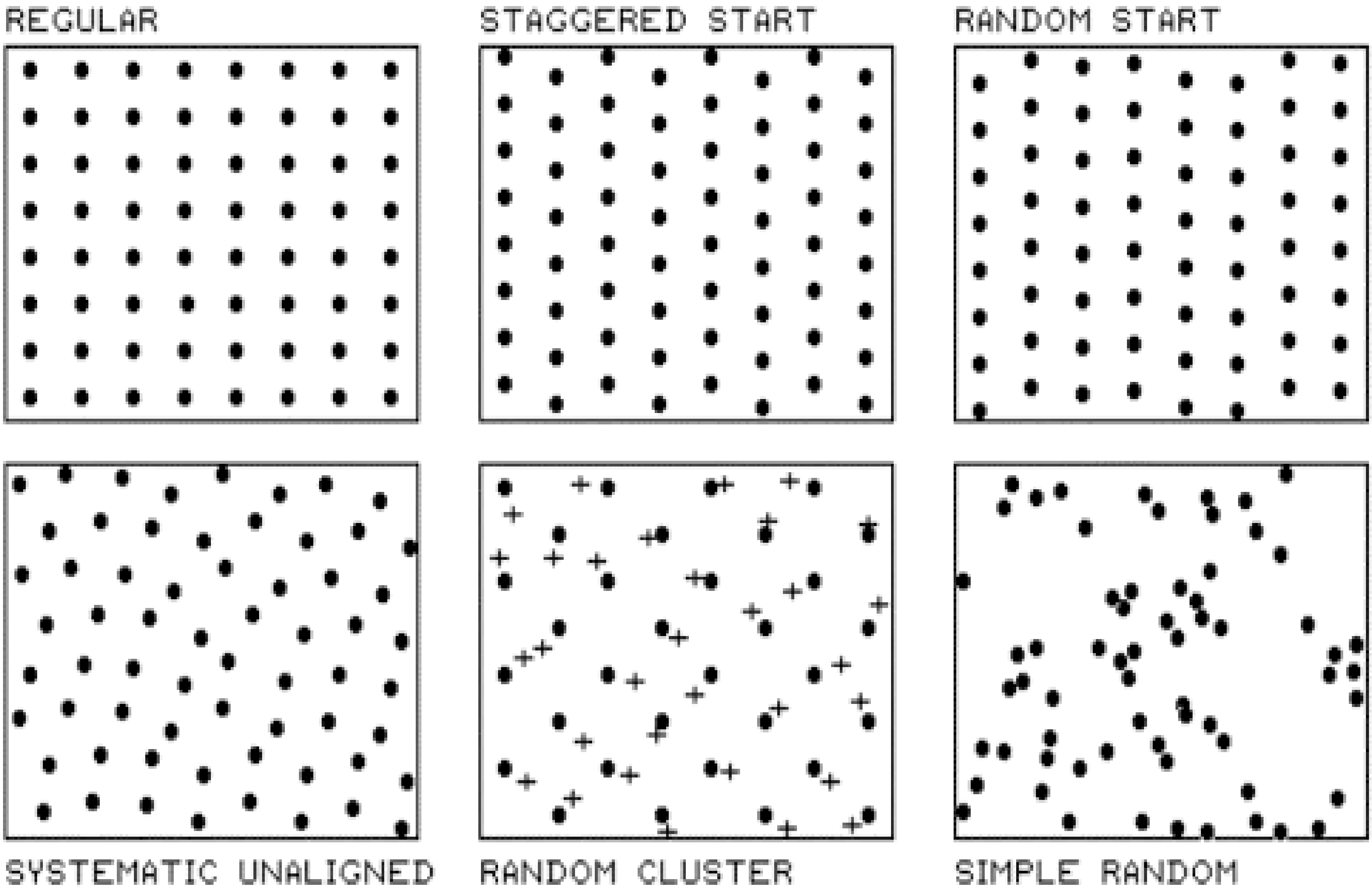
Box-Muller (10,000 runs)
p-value = 0.156 (K-S)
0
500
1000
1500
-4
-2
0
2
4
x
correlation(X,Y)=-0.0039
p-value = 0.83 (K-S)
0
500
1000
1500
-4
-2
0
2
4
y

Scatter Plot of Box-Muller



Bivariate Normal

二維觀察值的可能特性



https://lh3.googleusercontent.com/proxy/Y_bqJ5dNr3BMtfYSiabax1jiqY3MuwnHxVK590HoXbemgZsAaFThDRiivGT_qp_jOS2E4JikO1eYm-yFgmVqfDvIjXWqleUQ4U-lWMJ8oSNO_6E

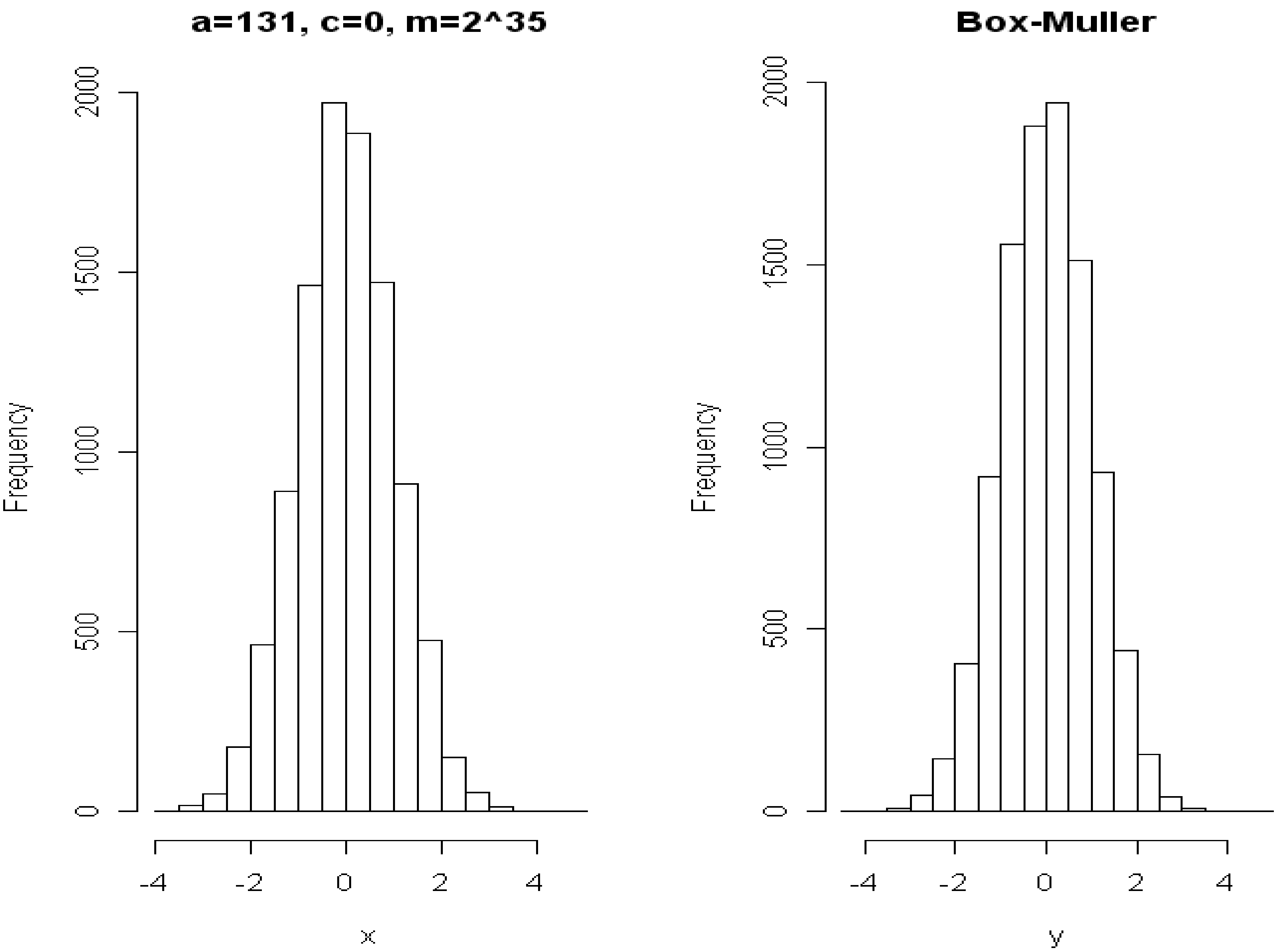
- Note: Using Box-Muller method with congruential generators must be careful. It is found by several researchers that

$$a\ (乘數) = 131$$

$$c\ (增量) = 0$$

$$m\ (除數) = 2^{35}$$

would have $X \in (-3.3, 3.6)$. See Neave (1973) and Ripley (1987, p.55) for further information.



I use 123,456 and 3,456 as seeds but did not find problems.

- Polar Method (recommended!)

→ Rejection method for generating two independent normal variables.

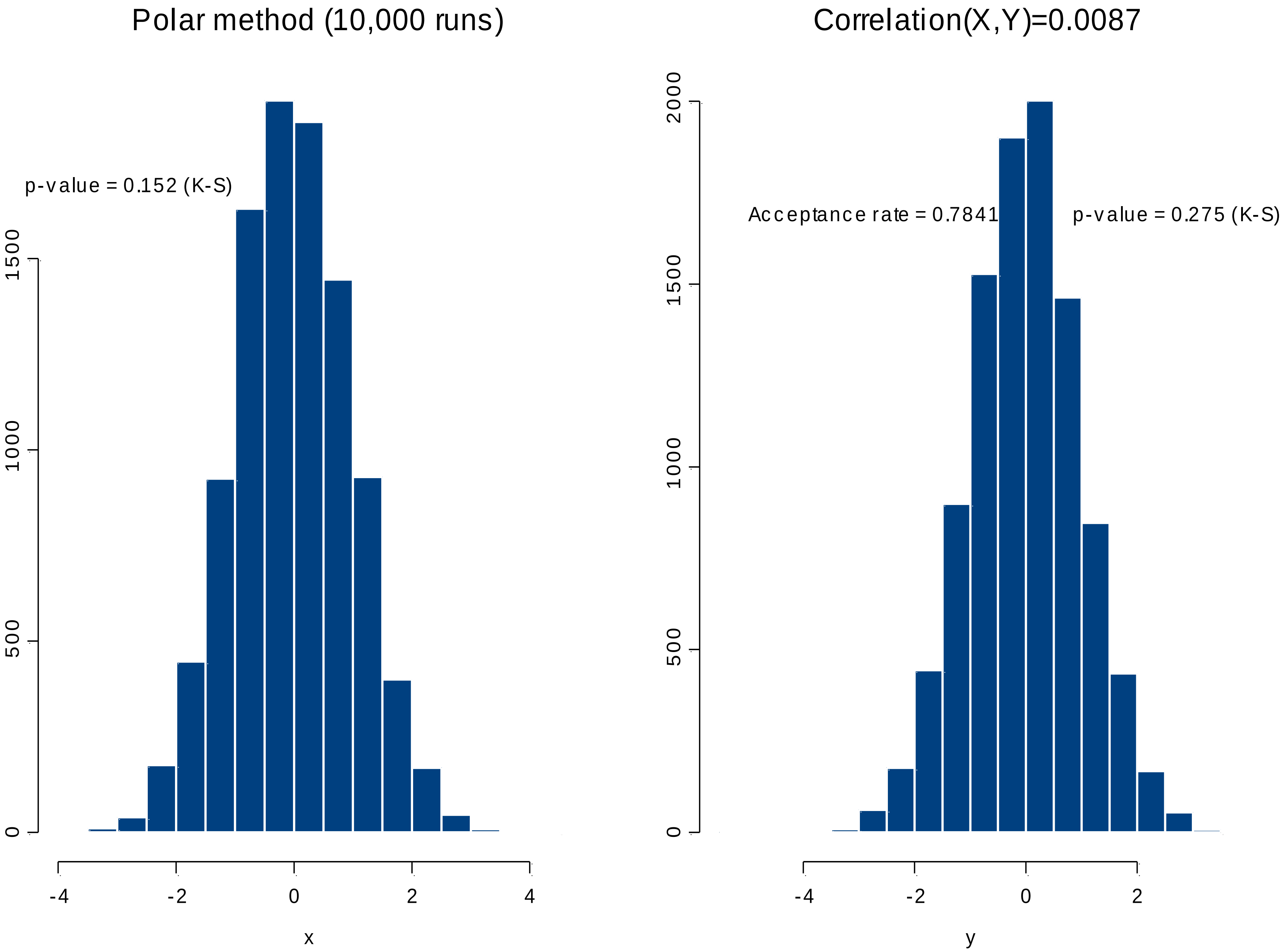
- Algorithm

1. Generate $V_1, V_2 \sim U(-1,1)$

   retain if $W = V_1^2 + V_2^2 < 1$

2. Let $C = \sqrt{-2W^{-1}\log W}$

3. Then $X = CV_1$ *and* $Y = CV_2$ are independent standard normal variables.



Polar method (10,000 runs)
p-value = 0.152 (K-S)
0
500
1000
1500
-4
-2
0
2
4
x
Correlation(X,Y)=0.0087
Acceptance rate = 0.7841
p-value = 0.275 (K-S)
0
500
1000
1500
2000
-4
-2
0
2
y

- Ratio-of-uniforms <u>(recommended!)</u>

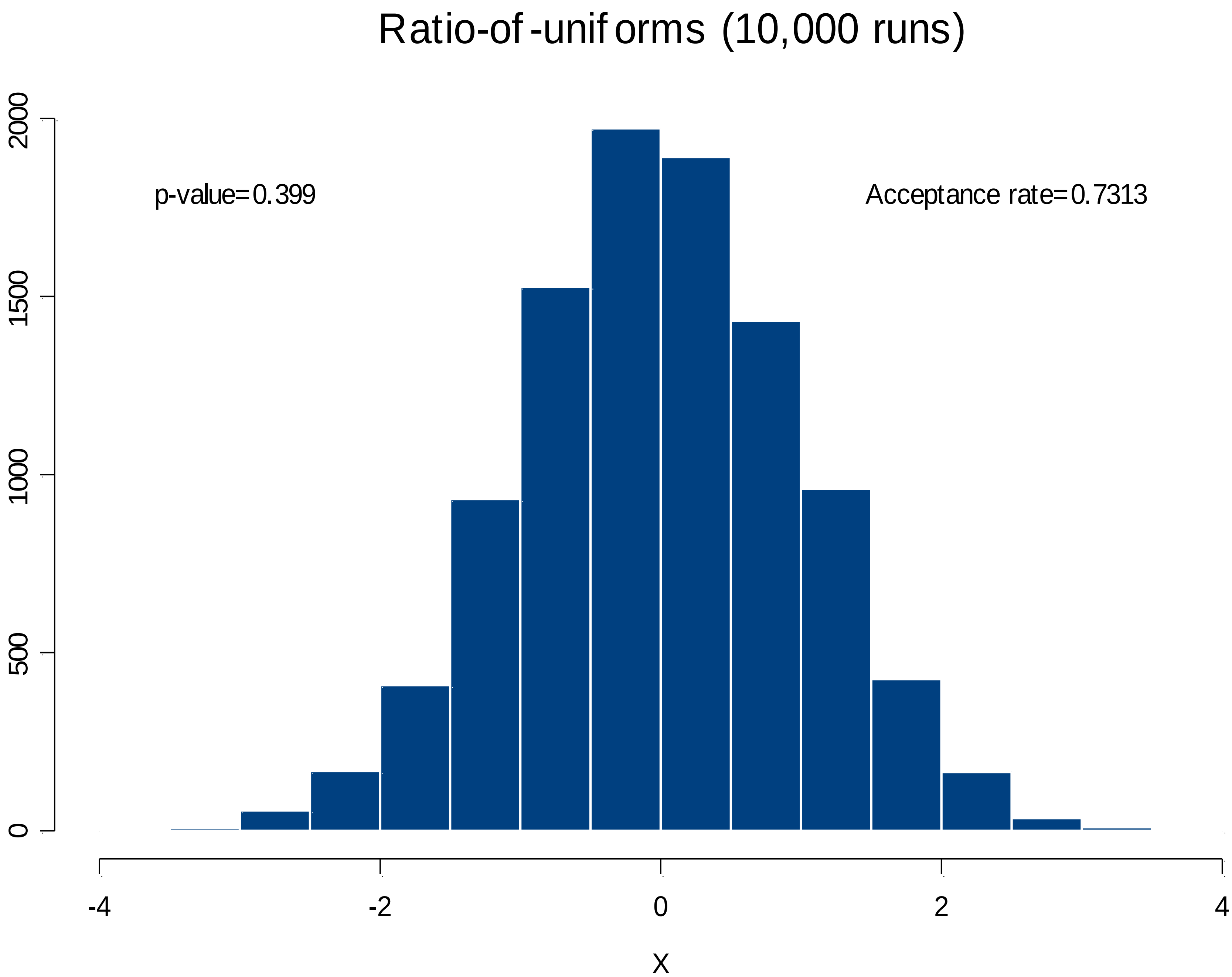
→Similar to Polar method, Ratio-of-uniforms is a rejection method.

- Algorithm

1. Generate $U_1, U_2 \sim U(0,1)$,

   and let $V = \sqrt{2/e}\,(2U_2 - 1)$

2. Let $X = V / U_1$, $Z = X^2/4$
3. Retain if $Z < 1 - U_1$ → Suggest deleting this step!
4. Retain if $Z \le 0.259 / U_1 + 0.35$ & $Z \le -\log U_1$
5. Then X is normally distributed.



Ratio-of-uniforms (10,000 runs)
p-value=0.399
Acceptance rate=0.7313
0
500
1000
1500
2000
-4
-2
0
2
4
X

- 問題：隨機變數有不同產生方法時，你/妳會選擇哪一種？(Choices of Generation) (有哪些判斷標準可供參考？)

→產生效率(e.g.成功接受率)？

→方法的複雜/方便程度？

→方法是否有瑕疵？(例如：與線性同餘法配合時是否有瑕疵？)

→取得的方便/廣為接受的方法？

→如何綜合判斷及考量？

- 是否有需要開發新的隨機變數產生方法？

→除非有特別需求(例如：學術研究、軟體配合等)，或是目標的隨機變數較為特殊，建議採用已經廣為大家認可的方法，可省卻驗證方法是否有效，也較不易遭受質疑。

註：如果其他軟體可以產生需要的變數，或者直接由該軟體輸出亂數，或者參考該軟體產生亂數的語法。(但可能會遭遇問題！)

- Ratio of Uniform (Cauchy Distribution)

→Ratio-of-uniforms can also be used to create random variables from Cauchy Distribution.

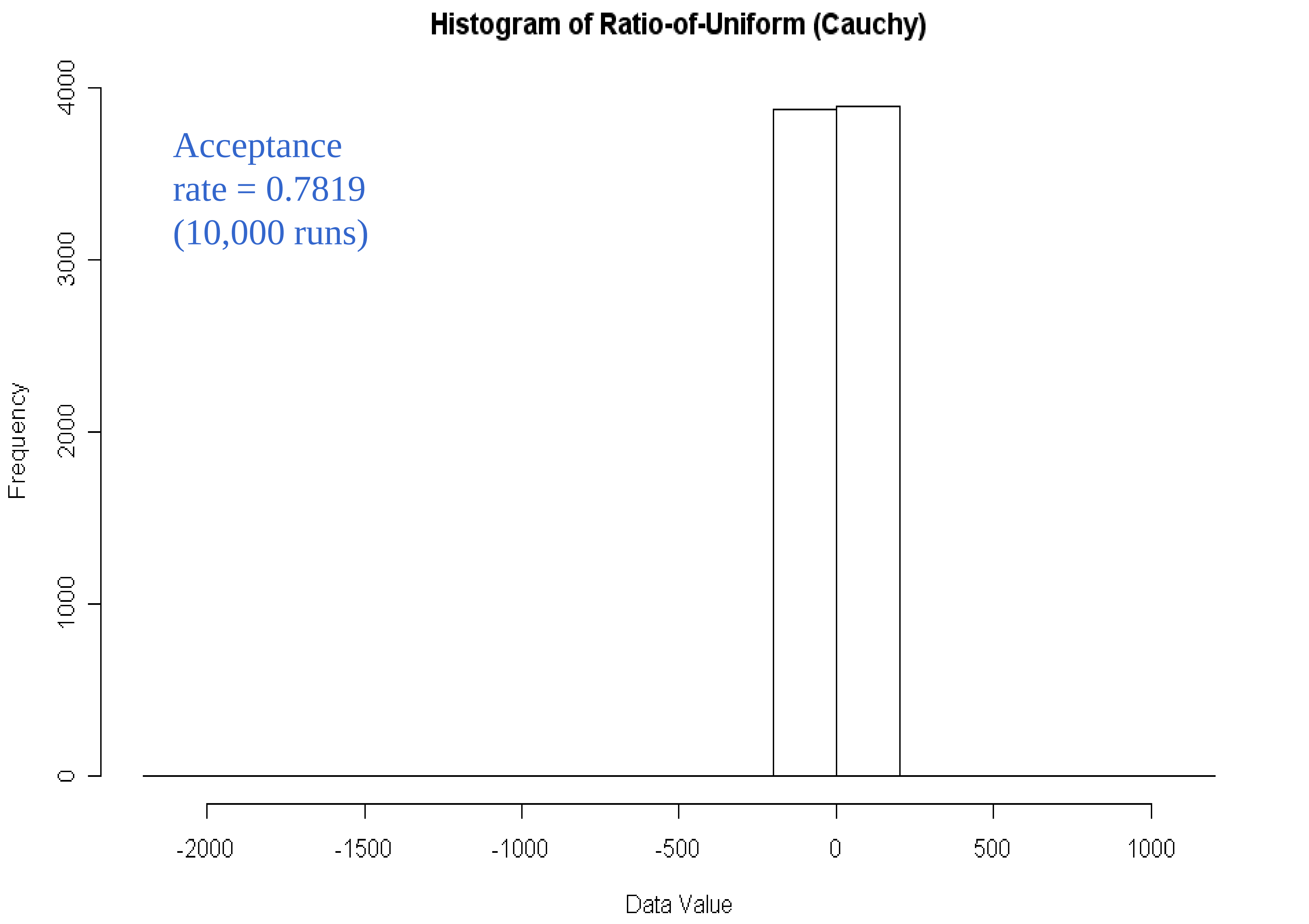
- Algorithm

1. Generate $U_1, U_2 \sim U(0,1)$, *and* $V = 2U_2 - 1$ retain if $U_1^2 + V^2 < 1$.
2. Then $X = V / U_1 \sim Cauchy(0,1)$, *i.e.*

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}.$$

e.g. 10,000 runs → p-value=0.9559

(Acceptance rate = 0.7819)



Histogram of Ratio-of-Uniform (Cauchy)
Acceptance
rate = 0.7819
(10,000 runs)
Frequency
0
1000
2000
3000
4000
-2000
-1500
-1000
-500
0
500
1000
Data Value

Inverse Transformation Method

Generates a random number from a probability density function by solving the probability. Density function's variable in terms of randomly generated numbers. This is achieved as follows:

- We solve the inverse of the integral of our probability density function at an arbitrary point a F(*a*), in terms of a random number *r*.
- We generate a unique random variable *a*, as follows: $a = F^{-1}(r)$. The foundation of this method is $F(X) \sim U(0,1)$ for all *X*.

- Note: Usually, we only use Inversion to create simpler r.v.'s, such as exponential distribution (i.e. $Exp(\lambda) = -\lambda \log(U(0,1))$. Although Inversion is a universal method, it may be too slow (unless subprograms to calculate F^{-1} are available).

→ In other words, although theoretically it is possible to create any r.v.'s, usually there are simpler methods.

- Example 1. $F(x) = x^2$, $0 < x < 1$ ➔ $X = U^{1/2}$.

- Example 2. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} F(x)$

  and we want to generate the minimum and maximum of X's.

→ Since $Y_n = \max(X_1, \ldots, X_n)$ & $Y_1 = \min(X_1, \ldots, X_n)$

which means $F_{y_n}(y) = [F(y)]^n$ & $F_{y_1}(y) = 1 - [1 - F(y)]^n$,

we can generate these two r.v.'s by

$$Y_n = F^{-1}(U^{1/n}) \;\&\; Y_1 = F^{-1}(1 - U^{1/n}).$$

$$Or, \quad Y_n = U^{1/n} \;\&\; Y_1 = 1 - U^{1/n} \;\; if \;\; X \sim U(0,1).$$

- Example 3. Generate $X \sim Poisson(\lambda)$, i.e.

$$F(i-1) = P(X \le i-1) < u \le P(X \le i) = F(i),$$

where $i \in \{0, 1, 2, \ldots\}$.

- Algorithm:

1. Generate $U_1 \sim U(0,1)$ and let $i = 0$.
2. If $U_1 \ge F(i)$, let $i = i + 1$;

   Otherwise $X = i$.

Note: This method can be used to generate any discrete distributions.

Notes:

(1) If total number of classes $\geq$ 30, we start from the middle (or mode).

(2) The expected number of trials is $E(X)+1$.

(3) We can use "*Indexed Search*" to increase the efficiency:

→Fix m, let $q_j = \min\{i \mid F(i) \geq \frac{j}{m}\},\ j = 0, \ldots, m-1$

Step 1. Generate $U \sim U(0,1)$, let $k=[mU]$ & $i=q_k$

Step 2. If $U \geq F(i)$, let $i = i + 1$;

Otherwise $X = i$.

- Example 4. *X ~ Poisson(10)*

(1) Usual method: 11 comparisons on average

(2) Mode: Reduced to about 3.54 comparisons

(3) Indexed search: we choose *m* = 5, i.e.

$$q_0 = 0, q_1 = 7, q_2 = 9, q_3 = 11, q_4 = 13.$$

Under 10,000 simulation runs (S-Plus), I found that one check on the table plus 2.346 comparisons ➔ 3.346

(Comparing to 3.3 in Ripley's book)

Composition Method

- We can generate complex distribution from simpler distributions, i.e.

$$F(x) = \sum_{i=1}^{m} \alpha_i \, F_i(x),$$

where $F_i(x)$ are d.f. of other variables.

→Or, from a conditional distribution,

$$f(x) = \sum_{i} p_i \; g(x \mid y = i)$$

$$f(x) = \int g(x \mid y) \, dF_Y(y).$$

- Example 1. To simulate x, where

$$P(x=i)=\begin{cases}0.05, & i=1,2,3,4,5\\ 0.15, & i=6,7,8,9,10\end{cases}$$

We independently generate $x_1 \in U\{1,2,...,10\}$ *and* $x_2 \in U\{6,7,...,10\}$. Let $X=X_1$ *or* X_2 *with probability 0.5.*

- Example 2. $X \sim B(5,0.2)$

→ *Sampling on each digit:*

$$P(X=i)=\begin{cases}0.3277, & i=0\\ 0.4096, & i=1\\ 0.2048, & i=2\\ 0.0512, & i=3\\ 0.0064, & i=4\\ 0.0003, & i=5\end{cases}$$

- Table method: (Composition)

Example 1. $X \sim B(3,1/3)$

	位置	10^{-1}	10^{-2}	10^{-3}
$P(X=0)=0.296$		2	9	6
$P(X=1)=0.445$		4	4	5
$P(X=2)=0.222$		2	2	2
$P(X=3)=0.037$		0	3	7
	總數	8	18	20

- Algorithm:

1. Generate U ~ U(0,1)
2. If $0 \le U < 0.8$, let $I = [10U]+1$, $X = a_1[I]$

   $0.8 \le U < 0.98$, let $I = [100U]-80+1$, $X = a_2[I]$

   $0.98 \le U$, let $I = [1000U]-980+1$, $X = a_3[I]$

其中

$a_1[.]$ = 00 1111 22

$a_2[.]$ = 000000000 1111 22 333

$a_3[.]$ = 00000 11111 22 3333333

- Example 3. Generate an r.v. *X* from

$$f(x) = n\int_1^{\infty} y^{-n} e^{-xy} dy, \quad 0 < x < \infty,$$

$$where\ dF_Y(y) = \frac{n\,dy}{y^{n+1}}, 1 < y < \infty, n \geq 1.$$

- Algorithm:

1. Generate U_1, U_2 from $U(0,1)$
2. Let $Y \leftarrow (U_1)^{-1/n}$
3. Return $X \leftarrow -log(U_2)/y$

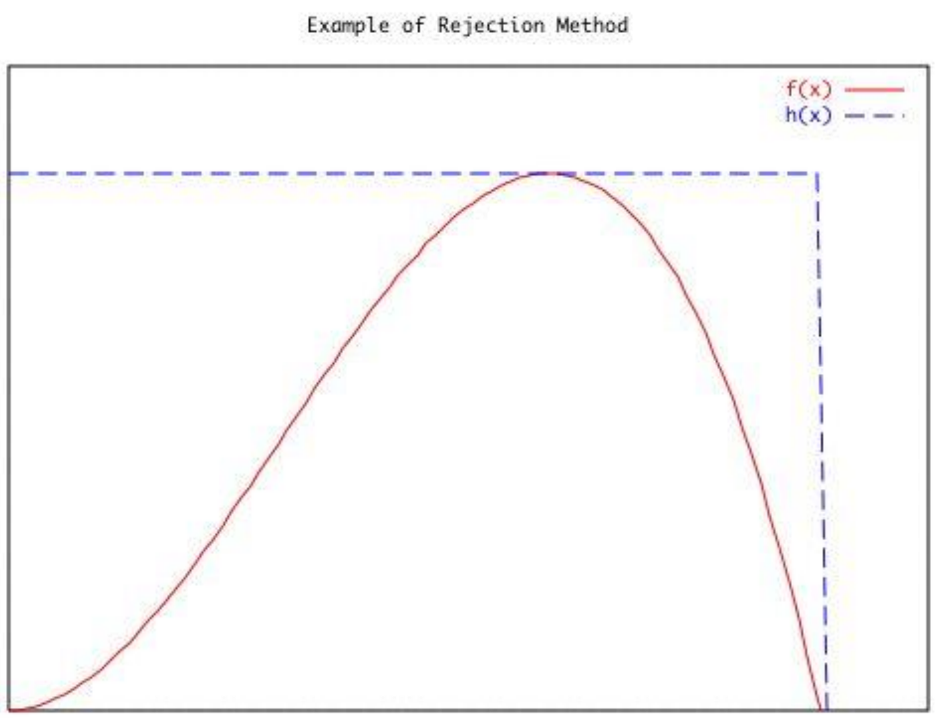
$$Note: \quad dF_Y(y) = \frac{n\,dy}{y^{n+1}} \Rightarrow F_Y(y) = \frac{1}{y^n} \sim U(0,1).$$

Rejection Method

Generates random numbers for a distribution function f(x). This is achieved as follows:

- Define a comparison function h(x) such that it encloses the desired function f(x).
- Choose uniformly distributed random points under h(x).
- If a point lies outside the area under f(x) reject it and choose another point.

Illustration of the Rejection Method



The following is an illustration of the rejection method using a square function for the comparison function.

- Example 1. X ~ Beta(2,4), i.e. $E(X)=1/3$,

$$f(x) = 20x(1-x)^3, \quad 0 < x < 1.$$

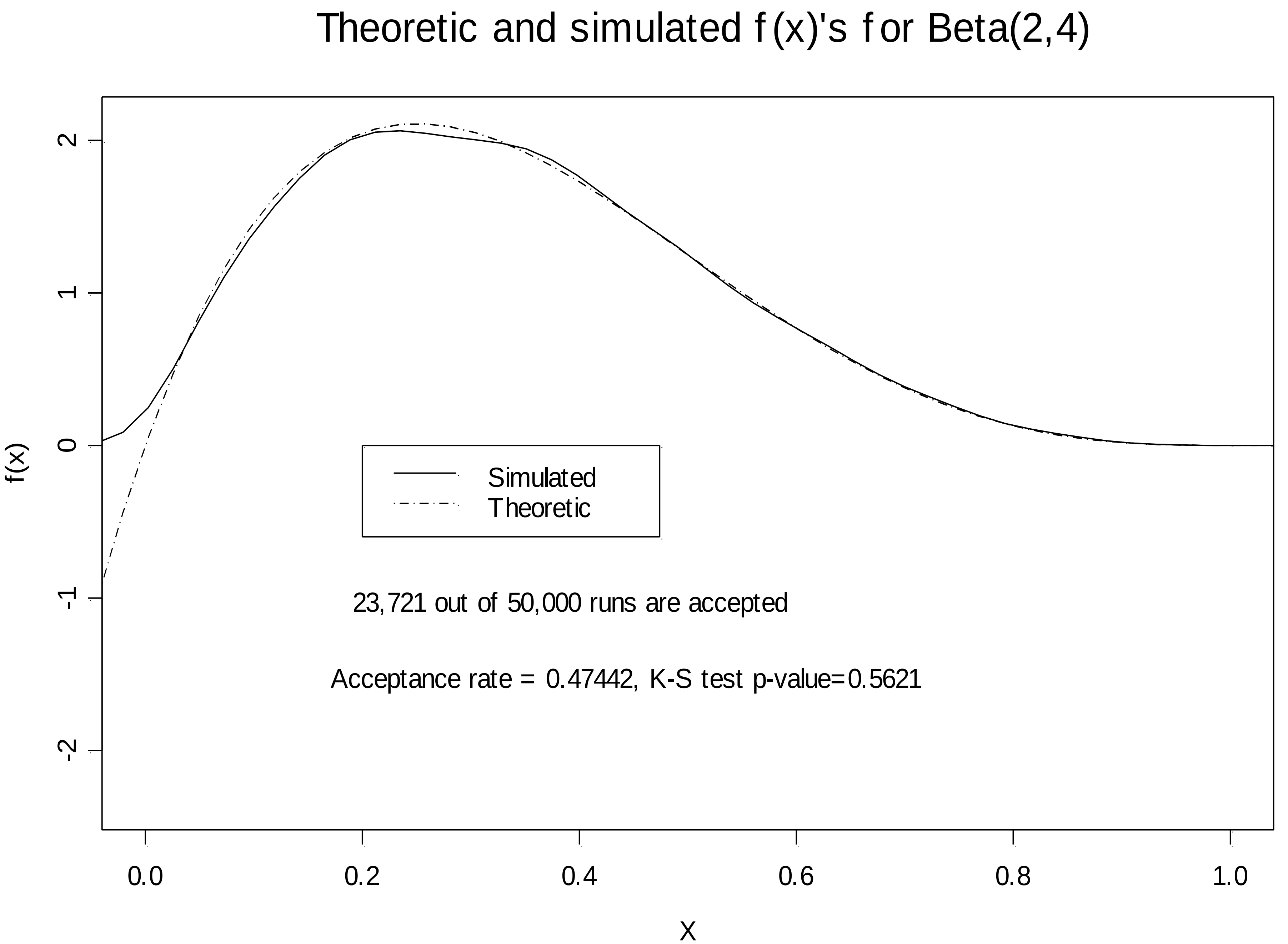
Then we use $g(x)=1$, $0 < x < 1$ as "envelop" to create $f(x)$.

Note: $$\frac{f(x)}{g(x)} = 20x(1-x)^3 \leq \frac{135}{64} = C \approx (0.4741)^{-1}.$$

- Algorithm:

1. Generate X, $U_1 \sim U(0,1)$.

2. If $U_1 \leq \dfrac{f(x)}{C \cdot g(x)}$ return Y = X. (Q: Rejection rate?)


Theoretic and simulated f(x)'s for Beta(2,4)
f(x)
X
2
1
0
-1
-2
0.0
0.2
0.4
0.6
0.8
1.0
Simulated
Theoretic
23,721 out of 50,000 runs are accepted
Acceptance rate = 0.47442, K-S test p-value=0.5621

- Example 2. Generate *Gamma(3/2,1), i.e.*

$$f(x) = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} e^{-x}, x > 0.$$

We want to generate *X* from $g(x) = \frac{2}{3} e^{-2x/3}$.

The max. of *f(x)/g(x)* is obtained when

$$\frac{1}{2} x^{-1/2} e^{-x/3} = \frac{1}{3} x^{1/2} e^{-x/3} \Rightarrow x = \frac{3}{2},$$

since $\frac{f(x)}{g(x)} = \frac{3}{\sqrt{\pi}} x^{1/2} e^{-x/3}$.

$$\therefore\ C = \frac{3}{\sqrt{\pi}} \left(\frac{3}{2}\right)^{1/2} e^{-1/2} = \frac{3\sqrt{3}}{\sqrt{2\pi e}} \cong 1.257317.$$

- Algorithm:

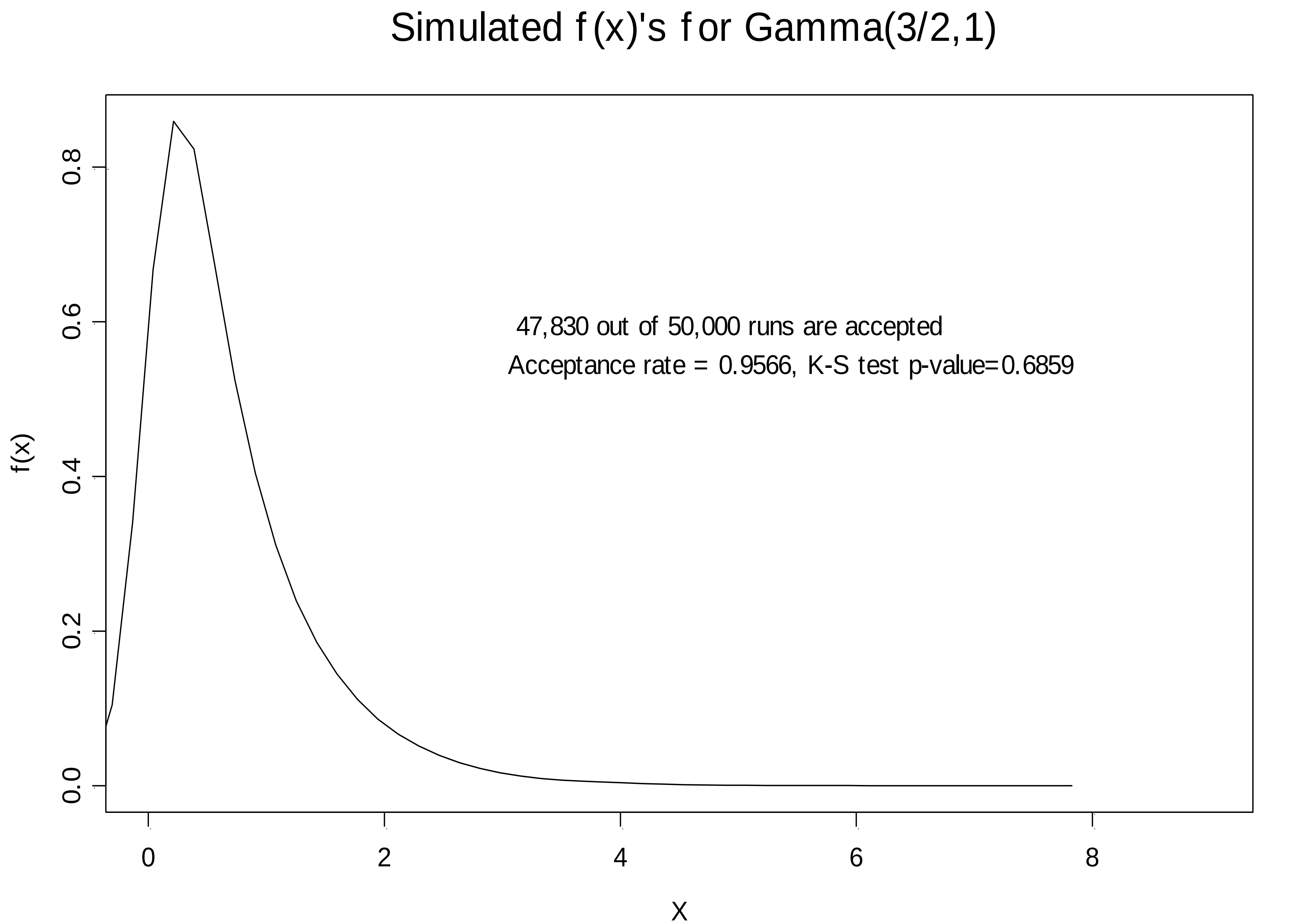
1. Generate U, $U_1 \sim U(0,1)$

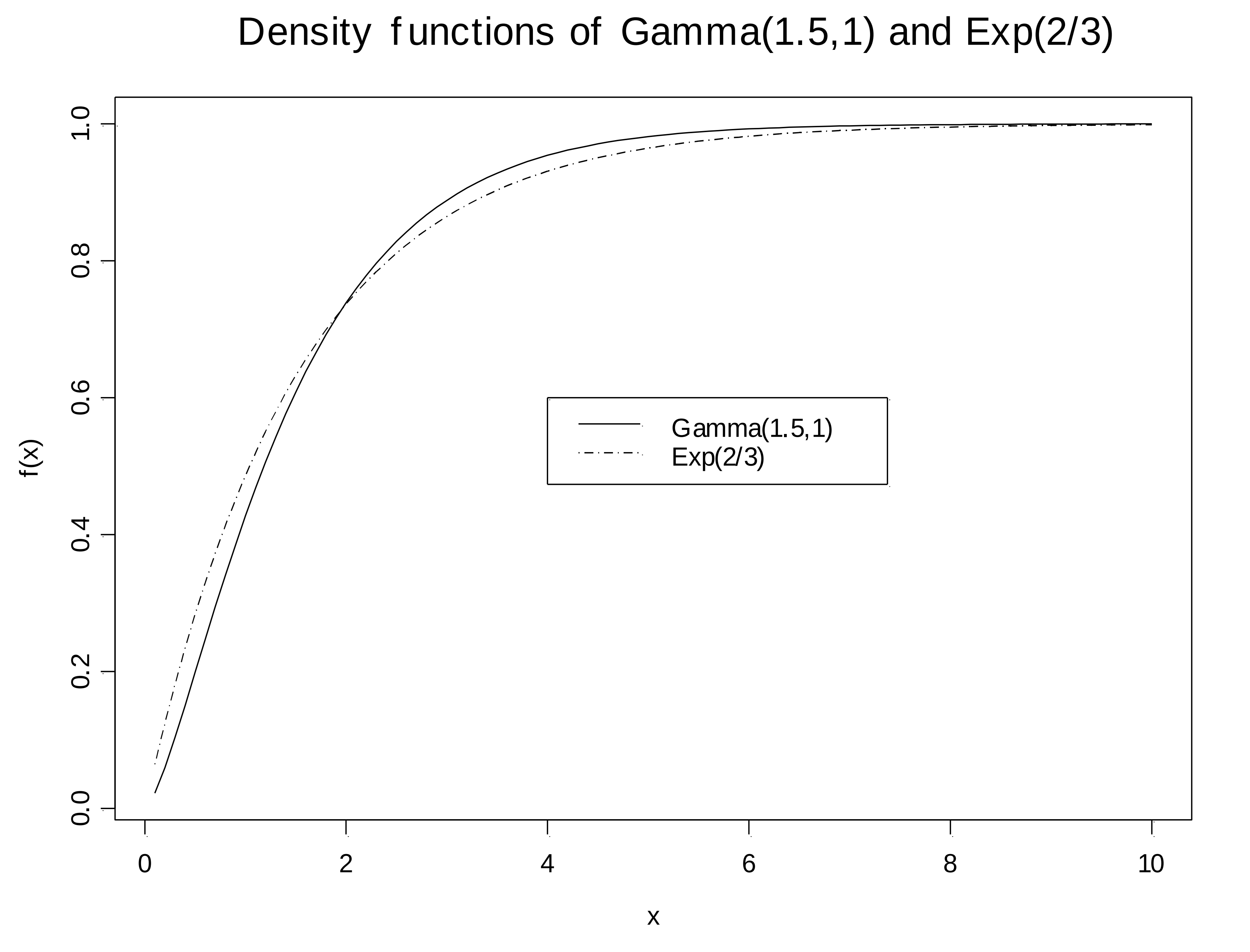
   Let $X = -3\log U/2$.

2. Return $Y = X$ if $U_1 \leq \dfrac{f(y)}{C \cdot g(y)}$.

- Question: Why do we choose $Y \sim Exp(2/3)$?

→ $Gamma(3/2,1)$ and $Exp(2/3)$ have the same mean!


Simulated f(x)'s for Gamma(3/2,1)
47,830 out of 50,000 runs are accepted
Acceptance rate = 0.9566, K-S test p-value=0.6859
f(x)
0.0
0.2
0.4
0.6
0.8
0
2
4
6
8
X


Density functions of Gamma(1.5,1) and Exp(2/3)
1.0
0.8
0.6
0.4
0.2
0.0
f(x)
0
2
4
6
8
10
x
Gamma(1.5,1)
Exp(2/3)

- Alias method: Looks like "rejection" but it is indeed "composition".

- Example 1. $X \sim B(3,1/3)$, i.e.

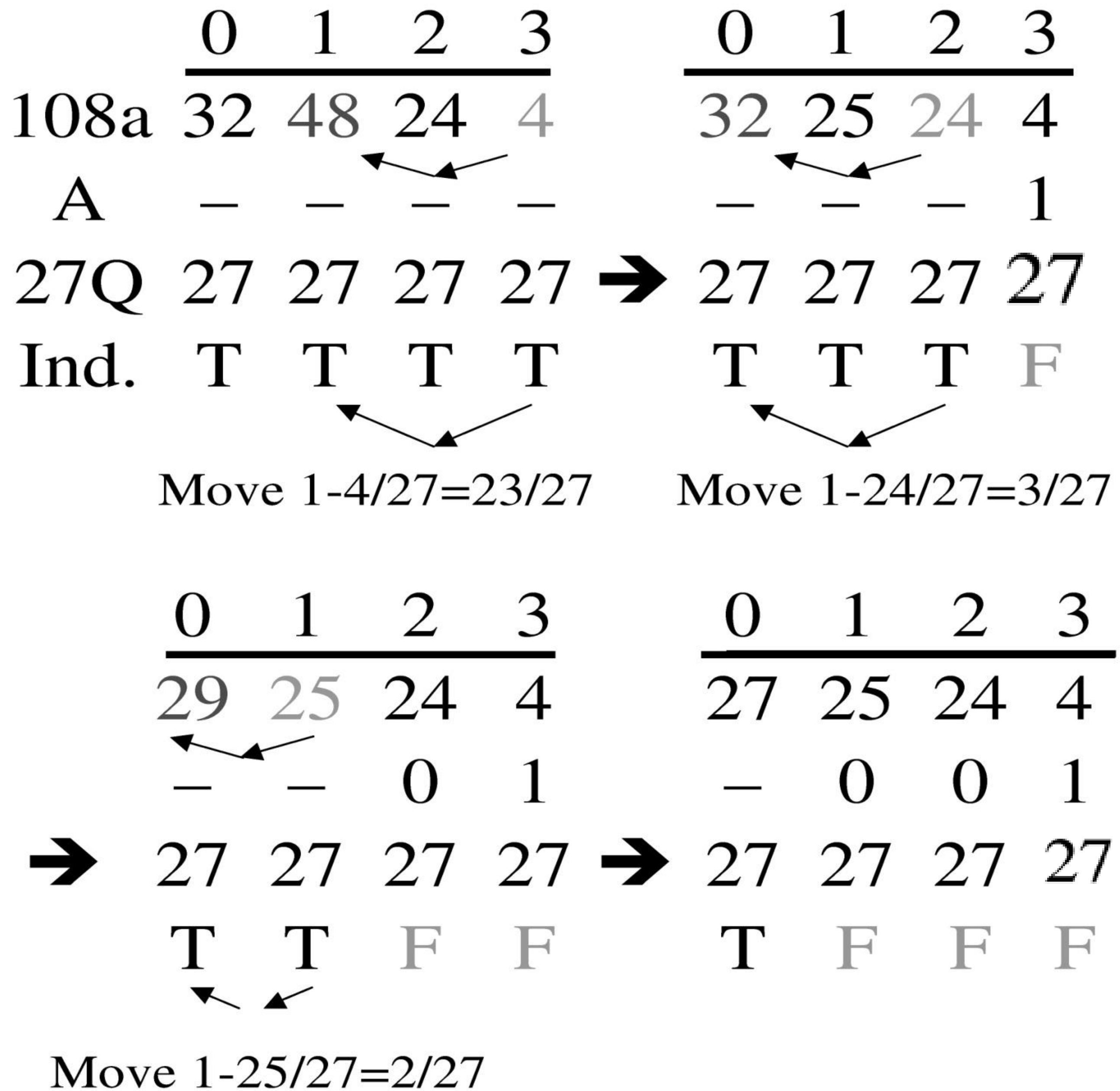
$$P(X=i)=\frac{8}{27},\frac{12}{27},\frac{6}{27},\frac{1}{27}\ for\ i=0,1,2,3.$$

$$P(X=0)=\frac{1}{4}+\left[\frac{2/27}{4}+\frac{3/27}{4}\right]=\frac{32}{108};$$

$$P(X=1)=\frac{25}{108}+\left[\frac{23/27}{4}\right]=\frac{48}{108};$$

$$P(X=2)=\frac{24}{108};$$

$$P(X=3)=\frac{4}{108}.$$


0 1 2 3
108a 32 48 24 4
A – – – –
27Q 27 27 27 27
Ind. T T T T
Move 1-4/27=23/27
0 1 2 3
32 25 24 4
– – – 1
27 27 27 27
T T T F
Move 1-24/27=3/27
0 1 2 3
29 25 24 4
– – 0 1
27 27 27 27
T T F F
Move 1-25/27=2/27
0 1 2 3
27 25 24 4
– 0 0 1
27 27 27 27
T F F F