

Section 6.6

2. Each of these solutions rests on additivity over integrals and the First Fundamental Theorem:

$$\text{a. } F(x) = \begin{cases} \int_1^x 2 \, dt & \text{if } 1 \leq x \leq 3 \\ F(3) + \int_3^x 6 \, dt & \text{if } 3 < x \leq 4 \end{cases} = \begin{cases} 2(x-1) & \text{if } 1 \leq x \leq 3 \\ 4 + 6(x-3) & \text{if } 3 < x \leq 4 \end{cases} = \begin{cases} 2x-2 & \text{if } 1 \leq x \leq 3 \\ 6x-14 & \text{if } 3 < x \leq 4 \end{cases}$$

$$\text{b. } F(x) = \begin{cases} \int_1^x t^2 \, dt & \text{if } 0 \leq x \leq 1 \\ F(1) + \int_1^x t \, dt & \text{if } 1 < x \leq 2 \end{cases} = \begin{cases} x^3/3 & \text{if } 0 \leq x \leq 1 \\ \frac{1}{3} + \frac{x^2-1}{2} & \text{if } 1 < x \leq 2 \end{cases} = \begin{cases} x^3/3 & \text{if } 0 \leq x \leq 1 \\ \frac{x^2}{2} - \frac{1}{6} & \text{if } 1 < x \leq 2 \end{cases}$$

$$\text{c. } F(x) = \begin{cases} \int_1^x t \, dt & \text{if } -1 \leq x < 0 \\ F(0) + \int_0^x t+1 \, dt & \text{if } 0 \leq x \leq 1 \end{cases} = \begin{cases} (x^2-1)/2 & \text{if } -1 \leq x < 0 \\ -1/2 + (x+1)^2/2 - 2 & \text{if } 0 \leq x \leq 1 \end{cases}$$

4. Since the proposed equation is true when $x=0$, it will suffice by the Identity Criterion to prove that the left- and right-hand sides of this equation have the same derivative for all x .

For this we rewrite the right-hand side as $f(0) + f'(0)x + x \int_0^x f''(t) \, dt - \int_0^x t f''(t) \, dt$, so that

its derivative is $f'(0) + \cancel{xf''(x)} + \int_0^x f''(t) \, dt - \cancel{xf''(x)} = f'(0) + \int_0^x f''(t) \, dt$ by the product rule and the Second Fundamental Theorem. But the continuity of f'' implies that

$\int_0^x f''(t) \, dt = f'(x) - f'(0)$ by the First Fundamental Theorem; thus the derivative of the right-hand side is $f'(x)$ for all x , completing the proof.

6. First, since the function $f: [1, \infty) \rightarrow \mathbb{R}$ defined by $f(t) = 1/(2\sqrt{t}-1)$ is continuous, it is integrable over the interval $[1, x]$ for any $x > 1$, so that F is defined on $[1, \infty)$; moreover F is continuous there by Proposition 6.27. Next we show that F is unbounded on $[1, \infty)$. Indeed, the monotonicity of the integral and the First Fundamental Theorem imply that

$$F(x) = \int_1^x 1/(2\sqrt{t}-1) \, dt \geq \int_1^x 1/2\sqrt{t} \, dt = \sqrt{x} - 1; \text{ since } \sqrt{x} - 1 \text{ is unbounded on } [1, \infty), \text{ we}$$

conclude that F is as well. It follows that given $c > 0$, we can find $x_0 > 1$ with $F(x_0) > c$, and since $F(1) = 0$, the Intermediate Value Theorem implies that the equation $F(x) = c$ has a solution $x \in [1, x_0]$. Finally, the Second Fundamental Theorem implies that

$F'(x) = f(x) = 1/(2\sqrt{x}-1) > 0$ for all $x > 1$, so that F is increasing on $[1, \infty)$, meaning that the aforementioned solution is unique.

8. Our assumption gives $\int_0^1 p(t) \, dt = \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = 0$. Since p is a polynomial, and therefore continuous, by the Mean Value Theorem for Integrals (as strengthened in Exercise 9, below) there exists a point $x \in (0, 1)$ such that $p(x) = (1/1) \int_0^1 p = 0$.

10. Let $\varepsilon > 0$. By our assumption that f is continuous at x_0 , we can find a positive number δ such that $|f(x) - f(x_0)| < \varepsilon/2$, or $f(x_0) - \varepsilon/2 < f(x) < f(x_0) + \varepsilon/2$, if $|x - x_0| < \delta$. If $x_0 < x < x_0 + \delta$, then it follows from the Monotonicity Property that

$[f(x_0) - \varepsilon](x - x_0) < \int_{x_0}^x f < [f(x_0) + \varepsilon](x - x_0)$; on the other hand, if $x_0 - \delta < x < x_0$, then

$[f(x_0) - \varepsilon](x_0 - x) < \int_x^{x_0} f < [f(x_0) + \varepsilon](x_0 - x)$, so that

$[f(x_0) - \varepsilon](x - x_0) > \int_{x_0}^x f > [f(x_0) + \varepsilon](x - x_0)$. In either case

$$f(x_0) - \varepsilon < \frac{F(x) - F(x_0)}{x - x_0} < f(x_0) + \varepsilon,$$

which shows that $F'(x_0) = \lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$, as desired.

12. First, the fact that $H(a) = 0$ follows from the fact that $\int_a^a \alpha f + \beta g$, etc. are by definition

zero. Next, the Second Fundamental Theorem and the definition of $\alpha f + \beta g$ give

$H'(x) = (\alpha f + \beta g)(x) - \alpha f(x) - \beta g(x) = 0$ for all x in (a, b) , so that by the Identity Crite-

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on there exists a number c with $H(x) = c$ for all x in (a, b) ; then as in the preceding exer-

cise we have in fact $H(x) = c$ for all x in $[a, b]$. Finally, taking $x = a$ shows that $c = 0$,

proving the linearity of the integral when the functions are continuous.