

## Section 6.5

2. a. The functions  $f:[1,2] \rightarrow \mathbb{R}$  and  $F:[1,2] \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{x^2} + x + \cos x$  and  $F(x) = -\frac{1}{x} + \frac{x^2}{2} + \sin x$  satisfy  $F'(x) = f(x)$ ; since  $f$  is continuous and bounded on  $(1,2)$ , the First Fundamental Theorem with  $a=1$  and  $b=2$  gives
- $$\int_1^2 f = F(2) - F(1) = (-1/2 + 2 + \sin 2) - (-1 + 1/2 + \sin 1) = 2 + \sin 2 - \sin 1 \approx 2.068.$$

- b. The functions  $f:[0,1] \rightarrow \mathbb{R}$  and  $F:[0,1] \rightarrow \mathbb{R}$  defined by  $f(x) = x\sqrt{4-x^2}$  and  $F(x) = -(4-x^2)^{3/2}/3$  satisfy  $F'(x) = f(x)$ ; since  $f$  is continuous and bounded on  $(0,1)$ , the First Fundamental Theorem with  $a=0$  and  $b=1$  gives

$$\int_0^1 f = F(1) - F(0) = -\frac{3^{3/2}}{3} + \frac{8}{3} = \frac{8-3^{3/2}}{3} \approx 0.935.$$

- c. The functions  $f:[1,3] \rightarrow \mathbb{R}$  and  $F:[1,3] \rightarrow \mathbb{R}$  defined by  $f(x) = x\sqrt{10-x}$  and  $F(x) = (2/5)(10-x)^{5/2} - (20/3)(10-x)^{3/2}$  satisfy  $F'(x) = f(x)$ ; since  $f$  is continuous and bounded on  $(1,3)$ , the First Fundamental Theorem gives

$$\int_1^3 f = F(3) - F(1) = \left(\frac{2}{5}7^{5/2} - \frac{20}{3}7^{3/2}\right) - \left(\frac{2}{5} - \frac{20}{3}\right) \approx 11.1883.$$

- d. It follows from the proof of Theorem 5.17 that  $\cos^2 x = (1 + \cos 2x)/2$  for all  $x$ . Therefore, the functions  $f:[0,\pi] \rightarrow \mathbb{R}$  and  $F:[0,\pi] \rightarrow \mathbb{R}$  defined by  $f(x) = (1 + \cos 2x)/2$  and  $F(x) = (2x + \sin 2x)/4$  satisfy  $F'(x) = f(x)$ ; since  $f$  is continuous and bounded on  $(1,3)$ , the First Fundamental Theorem gives

$$F(\pi) - F(0) = (2\pi + \sin 2\pi - [2 \cdot 0 + \sin(2 \cdot 0)]) / 4 = \pi / 2.$$

4. a. If indeed such an antiderivative  $F$ , then  $F'(x) = 4$  for all  $x \in (2,3)$ ; since the function  $4x$  also has this property, it follows from the Identity Criterion that  $F(x) = 4x + c_1$  for some number  $c_1$ . Moreover, since  $F$  is by definition continuous on  $[2,6]$ ,  $F(2) = \lim_{x \rightarrow 2, x > 2} F(x) = \lim_{x \rightarrow 2, x > 2} 4x + c_1 = 8 + c_1$ , so that  $F(x) = 4x + c_1$  for all  $x \in [2,3]$ . A parallel argument shows that there exists a number  $c_2$  with  $F(x) = c_2$  for all  $x \in [3,6]$ .

- b. The definition of antiderivative implies that  $F$  is continuous at  $x=3$ , so that  $12 + c_1 = \lim_{x \rightarrow 3, x < 3} F(x) = \lim_{x \rightarrow 3, x > 3} F(x) = c_2$ .

- c. For  $x \in [2,6]$  with  $x \neq 3$ , part b implies that the difference quotient  $\frac{F(x) - F(3)}{x - 3}$  is  $\frac{4x + c_1 - c_2}{x - 3} = \frac{4x - 12}{x - 3} = 4$  if  $x < 3$  and  $\frac{c_2 - 3}{x - 3}$  if  $x > 3$ . Thus  $\lim_{x \rightarrow 3, x > 3} \frac{F(x) - F(3)}{x - 3}$  exists only if  $c_2 = 3$ , and in this case  $\lim_{x \rightarrow 3, x > 3} \frac{F(x) - F(3)}{x - 3} = 0 \neq 4 = \lim_{x \rightarrow 3, x < 3} \frac{F(x) - F(3)}{x - 3}$ . Therefore regardless of  $c_1$  and  $c_2$ ,  $\frac{F(x) - F(3)}{x - 3}$  has no limit as  $x$  approaches 3, which is to say that  $F$  is not differentiable at  $x=3$ . It follows that  $f$  has no antiderivative.

6. Suppose that  $F$  satisfies the assumptions of Theorem 6.22. Then the function  $G: [a, b] \rightarrow \mathbb{R}$  defined by  $G(x) = F(x)$ ,  $a < x \leq b$  and  $G(a) = F(a) + 1$  is also differentiable on  $(a, b)$  with  $G': (a, b) \rightarrow \mathbb{R}$  both continuous and bounded, since  $G'(x) = F'(x)$  for all  $x \in (a, b)$ . It follows from Theorem 6.19 that  $G'$  is integrable on  $[a, b]$  with  $\int_a^b G'(x) dx = \int_a^b F'(x) dx$ . But if the requirement in Theorem 6.22 that  $F$  be continuous at (say) the endpoint  $x = a$  is unnecessary, then the theorem applies to  $G'$  to give

$$\int_a^b G'(x) dx = G(b) - G(a) = F(b) - F(a) - 1 \neq F(b) - F(a) = \int_a^b F'(x) dx;$$

the contradiction shows that the requirement is in fact necessary. The proof of the theorem uses this assumption in asserting the existence of the points  $c_1$  and  $c_n$ .