

## Section 6.4

2. Let  $P = \{x_0, \dots, x_n\}$  be any partition of  $[0, 1]$ . The definition of  $f$  and the density of both the rationals and irrationals give  $M_i = x_i$  and  $m_i = -x_i$  for all  $i = 1, \dots, n$ . Thus
 
$$U(f, P) - L(f, P) = \sum_{i=1}^n 2x_i(x_i - x_{i-1}).$$
 If  $P$  contains the point  $1/2$ , then we can write
 
$$U(f, P) - L(f, P) = \sum_{i=1}^{m-1} 2x_i(x_i - x_{i-1}) + \sum_{i=m}^n 2x_i(x_i - x_{i-1}),$$
 where  $x_m = 1/2$ ; since  $x_i \geq 1/2$  for  $i = m, \dots, n$ , the underlined summation is at least
 
$$2(1/2) \sum_{i=m}^n (x_i - x_{i-1}) = 2(1/2)(1 - 1/2) = 1/2.$$
 If instead  $P$  does not contain  $1/2$ , then we create the refinement  $P^* = P \cup \{1/2\}$ ; the preceding argument together with the Refinement Lemma then gives  $U(f, P) - L(f, P) \geq U(f, P^*) - L(f, P^*) \geq 1/2$ . Altogether,
 
$$U(f, P) - L(f, P) \geq 1/2$$
 for all partitions  $P$  of  $[0, 1]$ , so that by the Archimedes-Riemann Theorem  $f$  is not integrable.
  
4. The Archimedes-Riemann Theorem implies that there is an Archimedean sequence  $\{P_n\}$  of partitions for  $f$  of  $[a, b]$ , so that  $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$ . By definition of convergence, we can find a natural number  $N$  such that  $U(f, P_n) - L(f, P_n) < \varepsilon$  for all  $n \geq N$ ; therefore, we may take  $P_N$  to be the desired partition.
  
6. We suppose first that  $\int_0^1 f > 0$ . If there is no such point  $x_0$  as described in the conclusion, then our assumption implies that  $f(x) = 0$  for all  $x \in [0, 1]$ , meaning that  $\int_0^1 f = 0$ , a contradiction. Conversely, if  $f(x_0) > 0$  for some  $x_0 \in [0, 1]$ , then because  $f(x) \geq 0$  for all  $x \in [0, 1]$  we can argue as in the preceding exercise to find a positive number that is a lower bound for all upper Darboux sums of  $f$ , leading to  $\int_0^1 f > 0$ , as desired.
  
8. As in the proof of Theorem 6.19, we may choose  $M \geq 0$  such that  $-M \leq f(x) \leq M$  for all  $x \in [a, b]$ . Because  $x_0 \in (a, b)$ , we can choose sequences  $\{a_n\}$  in  $(a, x_0)$  and  $\{b_n\}$  in  $(x_0, b)$  such that  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = x_0$ . Fix an index  $n$ . Then the functions  $f: [a, a_n] \rightarrow \mathbb{R}$  and  $f: [b_n, b] \rightarrow \mathbb{R}$  are continuous, and hence integrable. Thus by the Archimedes-Riemann Theorem (together with Exercise 4, above), there are partitions  $P_n^*$  and  $P_n^{**}$  of  $[a, a_n]$  and  $[b_n, b]$ , respectively, such that  $0 \leq U(f, P_n^*) - L(f, P_n^*) < 1/n$  and  $0 \leq U(f, P_n^{**}) - L(f, P_n^{**}) < 1/n$ . We define  $P_n$  to be the partition of  $[a, b]$  obtained by appending  $x_0$  to the sets of partition points of  $P_n^*$  and  $P_n^{**}$ . Observe that

$$U(f, P_n) - L(f, P_n) = [U(f, P_n^*) - L(f, P_n^*)] + [U(f, P_n^{**}) - L(f, P_n^{**})] + A_n + B_n,$$

where  $A_n$  denotes the contribution to the difference of Darboux sums from the partition interval  $[a_n, x_0]$ , and  $B_n$  denotes the contribution to the difference of Darboux sums from the partition interval  $[x_0, b_n]$ . The fact that  $-M \leq f(x) \leq M$  for all  $x \in [a, b]$  implies that  $0 \leq A_n \leq 2M(x_0 - a_n)$  and  $0 \leq B_n \leq 2M(b_n - x_0)$ . Thus

$$\begin{aligned} & U(f, P_n) - L(f, P_n) \\ & \leq [U(f, P_n^*) - L(f, P_n^*)] + [U(f, P_n^{**}) - L(f, P_n^{**})] + 2M(x_0 - a_n) + 2M(b_n - x_0). \end{aligned}$$

However, by the choices of the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{P_n^*\}$ , and  $\{P_n^{**}\}$ ,

$$\lim_{n \rightarrow \infty} [U(f, P_n^*) - L(f, P_n^*)] = \lim_{n \rightarrow \infty} [U(f, P_n^{**}) - L(f, P_n^{**})] = \lim_{n \rightarrow \infty} (x_0 - a_n) = \lim_{n \rightarrow \infty} (b_n - x_0) = 0.$$

Therefore  $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$ . Thus,  $\{P_n\}$  is an Archimedean sequence of partitions for  $f$  on  $[a, b]$ . According to the Archimedes-Riemann Theorem, the function  $f$  is integrable on  $[a, b]$ .