

Section 6.2

2. We let $P_1 = \{x_0, x_1, \dots, x_n\}$ and $P_2 = \{y_0, y_1, \dots, y_m\}$. Because P_1 is a refinement of P_2 , each of the intervals $[x_{i-1}, x_i]$, $1 \leq i \leq n$ is contained in some interval $[y_{j-1}, y_j]$, where $1 \leq j \leq m$.

Thus, $x_i - x_{i-1} \leq y_j - y_{j-1}$, so that $\text{gap } P_1 = \max_{1 \leq i \leq n} [x_i - x_{i-1}] \leq \max_{1 \leq j \leq m} [y_j - y_{j-1}] = \text{gap } P_2$. On the other hand, the converse is false; as a counterexample we cite the partitions $P_1 = \{0, 2, 4, 6\}$ and $P_2 = \{0, 3, 6\}$ of $[a, b] = [0, 6]$; P_1 is not a refinement of P_2 , 3 is a partition point of P_2 but not of P_1 , and yet $\|P_1\| = 2 \leq 3 = \|P_2\|$.

4. a. This is shown in Example 1.1.

b. Because the function $f: [a, b] \rightarrow \mathbb{R}$ defined by $f(x) = x$ is monotone increasing, it is integrable by Example 6.9, with $\lim_{n \rightarrow \infty} U(f, P_n) - L(f, P_n) = 0$, where P_n denotes the regular partition of $[a, b]$ into n partition intervals; further, $M_i = f(x_i) = x_i$ for all i . By the Archimedes-Riemann Theorem, therefore, $\int_a^b x dx = \lim_{n \rightarrow \infty} U(f, P_n)$. However, part a and the definition of P_n give

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^n x_i \frac{(b-a)}{n} = \sum_{i=1}^n \left[a + i \frac{(b-a)}{n} \right] \frac{(b-a)}{n} = a \frac{(b-a)}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i \\ &= (b-a) \left[\frac{a}{n} + \frac{(b-a)}{n^2} \frac{n(n+1)}{2} \right] \rightarrow (b-a) \left[a + \frac{b-a}{2} \right] = (b-a) \frac{b+a}{2} = \frac{b^2 - a^2}{2} \end{aligned}$$

as $n \rightarrow \infty$; we conclude that $\int_a^b x dx = (b^2 - a^2)/2$, as desired.

6. a. This was shown in Exercise 4b, above.

b. We define $f: [a, b] \rightarrow \mathbb{R}$ by $f(x) = x^2$ and $P_n = \{x_0, \dots, x_n\}$, where $x_i = a + i(b-a)/n$, $i = 1, \dots, n$. Once again f is integrable because it is monotone increasing, with

$m_i = f(x_{i-1}) = x_{i-1}^2 = [a + (i-1)(b-a)/n]^2$ and $M_i = f(x_i) = x_i^2 = [a + i(b-a)/n]^2$ for all i . By Exercise 3 of Section 1.1,

$$\begin{aligned} U(f, P_n) &= \frac{b-a}{n} \sum_{i=1}^n \left(a + i \frac{b-a}{n} \right)^2 = \frac{b-a}{n} \sum_{i=1}^n \left(a^2 + 2a \frac{b-a}{n} i + \left(\frac{b-a}{n} \right)^2 i^2 \right) \\ &= (b-a) \left[a^2 + \frac{2a(b-a)}{n^2} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^2}{n^3} \frac{n(n+1)(2n+1)}{6} \right], \end{aligned}$$

which approaches

$$\begin{aligned} &\left[a^2 + a(b-a) + \frac{(b-a)^2}{3} \right] (b-a) = \left[ab + \frac{(b-a)^2}{3} \right] (b-a) \\ &= \left[ab + \frac{b^2 - 2ba + a^2}{3} \right] (b-a) = \left[\frac{b^2 + ba + a^2}{3} \right] (b-a) = \frac{b^3 - a^3}{3} \end{aligned}$$

as $n \rightarrow \infty$. Thus the Archimedes-Riemann Theorem implies that $\int_a^b x^3 = (b^3 - a^3)/3$, as desired.

8. By the Archimedes-Riemann Theorem, our assumption that f is integrable implies that there exists an Archimedean sequence $\{Q_n\}$ of partitions for f on $[a, b]$. We define a new sequence $\{P_n\}$ of partitions for f on $[a, b]$ by $P_n = Q_1 \cup Q_2 \cup \cdots \cup Q_n$, $n \in \mathbb{N}$, so that P_n is a refinement of Q_n and P_{n+1} is a refinement of P_n for all n . It follows from the Refinement Lemma that $L(f, Q_n) \leq L(f, P_n) \leq L(f, P_{n+1})$ and $U(f, P_{n+1}) \leq U(f, P_n) \leq U(f, Q_n)$ for all n , so that the sequence $\{L(f, P_n)\}$ is monotone decreasing and the sequence $\{U(f, P_n)\}$ is monotone increasing. Moreover $U(f, P_n) - L(f, P_n) \leq U(f, Q_n) - L(f, Q_n)$ for all n . Our assumption on $\{Q_n\}$ implies that $\lim_{n \rightarrow \infty} [U(f, Q_n) - L(f, Q_n)] = 0$, and so $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$ by the Comparison Lemma; thus $\{P_n\}$ is also an Archimedean

sequence of partitions for f on $[a, b]$. Finally, if $\{P_n\}$ is any sequence of partitions of $[a, b]$ satisfying the conditions of the exercise, then as above $L(f, P_n) \leq L(f, P_{n+1})$ and $U(f, P_{n+1}) \leq U(f, P_n)$ for all n , so that the sequences $\{U(f, P_n)\}$ and $\{L(f, P_n)\}$ are monotone decreasing and increasing, respectively.

10. For each $n \in \mathbb{N}$ we define the partition $P_n = \{x_0, \dots, x_{n+2}\}$ of $[2, 4]$ by $x_i = 2 + i/n$, $0 \leq i \leq n$, $x_{n+1} = 3 + 1/n$, and $x_{n+2} = 4$ for all n . The definition of f gives

$$L(f, P_n) = \sum_{i=1}^n \left(2 + \frac{i-1}{n}\right) \frac{1}{n} + 2 \frac{1}{n} + 2 \left(1 - \frac{1}{n}\right) = 2 + \frac{n-1}{2n} + 2 \frac{1}{n} + 2 \left(1 - \frac{1}{n}\right) = \frac{9}{2} - \frac{1}{n}$$

and

$$U(f, P_n) = \sum_{i=1}^n \left(2 + \frac{i}{n}\right) \frac{1}{n} + 3 \frac{1}{n} + 2 \left(1 - \frac{1}{n}\right) = 2 + \frac{n+1}{2n} + 3 \frac{1}{n} + 2 \left(1 - \frac{1}{n}\right) = \frac{9}{2} + \frac{2}{n}$$

for all n , so that $U(f, P_n) - L(f, P_n) = 3/n \rightarrow 0$ as $n \rightarrow \infty$; thus $\{P_n\}$ is an Archimedean sequence of partitions for f on $[2, 4]$, so that f is integrable by the Archimedes-Riemann Theorem.

12. First, by Exercise 13 in Section 3.1, f is continuous. As a result, the Extreme Value Theorem implies that f attains both a maximum and a minimum value on each partition interval $[x_{i-1}, x_i]$; we define $u_i, v_i \in [x_{i-1}, x_i]$ to be points where the maximum and minimum values, respectively, are attained. Thus $M_i - m_i = f(v_i) - f(u_i) \leq c|v_i - u_i| \leq c(x_i - x_{i-1})$ for each i , and so by the preceding exercise

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \leq c \sum_{i=1}^n (x_i - x_{i-1})^2 \leq c(b-a) \cdot \text{gap } P.$$