

Section 6.3

2. We define $p(\lambda)$ as in the hint, so that $p(\lambda) \geq 0$ for all λ by Theorem 6.13, because $(f - \lambda g)^2(x) \geq 0$ for all $x \in [a, b]$. But then our assumptions and Theorem 6.15 mean that

$$p(\lambda) = \int_a^b (f - \lambda g)^2 = \int_a^b f^2 - 2\lambda fg + \lambda^2 g^2 = \int_a^b f^2 - 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2 \geq 0,$$

so that $p(\lambda)$ is indeed a quadratic polynomial in λ whose discriminant therefore cannot be positive. It follows that $\left(2 \int_a^b fg\right)^2 - 4 \int_a^b g^2 \int_a^b f^2 \leq 0$, which gives the result.

4. We present the proofs for the supremum, the infimum cases being parallel. First we suppose that $\alpha > 0$. Now $x \leq \sup S$, and so $\alpha x \leq \alpha \sup S$, for all $x \in S$ by definition of supremum, so that $\sup \alpha S \leq \alpha \sup S$, again by definition of supremum. On the other hand, if $s = \sup \alpha S < \alpha \sup S$, then by definition of $\sup \alpha S$, $\alpha x \leq s < \alpha \sup S$ for all $x \in S$, so that

$x \leq \frac{s}{\alpha} < \sup S$ for all $x \in S$; this contradicts the definition of $\sup S$, and so we conclude that

$\sup \alpha S = \alpha \sup S$, as desired. Next, if $\alpha = 0$, then $\alpha S = \{0\}$, whose supremum is

$0 = \alpha \sup S$. Finally we suppose that $\alpha < 0$. Now $x \geq \inf S$, and so $\alpha x \leq \alpha \inf S$, for all $x \in S$ by definition of infimum, so that $\sup \alpha S \leq \alpha \inf S$ by definition of supremum. On the other hand, if $s = \sup \alpha S < \alpha \inf S$, then by definition of $\sup \alpha S$, $\alpha x \leq s < \alpha \inf S$ for all

$x \in S$, so that $x \geq \frac{s}{\alpha} > \inf S$ for all $x \in S$; this contradicts the definition of $\inf S$, and so we

conclude that $\sup \alpha S = \alpha \inf S$, as desired.

Now we prove the assertions in (6.31) dealing with upper sums, the lower sums again using a parallel argument. The definition of upper sum gives $U(\alpha f, P) = \sum_{i=1}^n M_i(\alpha f)(x_i - x_{i-1})$,

where $M_i(\alpha f)$ denotes $\sup\{(\alpha f)(x) \mid x \text{ in } [x_{i-1}, x_i]\}$, and $U(f, P) = \sum_{i=1}^n M_i(f)(x_i - x_{i-1})$, where $M_i(f)$ denotes $\sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}$; we similarly define $m_i(\alpha f)$ and $m_i(f)$.

First, if $\alpha > 0$, then the above results give

$$M_i(\alpha f) = \sup\{(\alpha f)(x) \mid x \text{ in } [x_{i-1}, x_i]\} = \alpha \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} = \alpha M_i(f), \quad i = 1, \dots, n,$$

and summing over all i gives $U(\alpha f, P) = \alpha U(f, P)$. Next, if $\alpha = 0$, then

$$U(\alpha f, P) = U(0 \cdot f, P) = U(0, P) = 0 = 0 \cdot U(f, P) = \alpha U(f, P),$$

so that $U(\alpha f, P) = \alpha U(f, P)$ whenever $\alpha \geq 0$. Finally, if $\alpha < 0$, then

$$M_i(\alpha f) = \sup\{(\alpha f)(x) \mid x \text{ in } [x_{i-1}, x_i]\} = \alpha \inf\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} = \alpha m_i(f), \quad i = 1, \dots, n,$$

and summing over i gives $U(\alpha f, P) = \alpha L(f, P)$, as desired.

6. In light of the Archimedes-Riemann Theorem, our assumption implies that there are Archimedean sequences of partitions $\{Q_n\}$ and $\{R_n\}$ for f over $[a, c]$ and $[c, b]$, respectively. It follows that given $\varepsilon > 0$, we can find natural numbers N_1 and N_2 such that

$$0 \leq U(f, Q_n) - L(f, Q_n) < \varepsilon/2 \text{ if } n \geq N_1 \text{ and } 0 \leq U(f, R_n) - L(f, R_n) < \varepsilon/2 \text{ if } n \geq N_2.$$

Now for each natural number n we define the partition P_n to be the union of the partition points of Q_n and R_n , so that $U(f, P_n) = U(f, Q_n) + U(f, R_n)$ and

$L(f, P_n) = L(f, Q_n) + L(f, R_n)$ for all n , by definitions of upper and lower sum. If the natural number $n \geq \max\{N_1, N_2\}$, then

$$U(f, P_n) - L(f, P_n) = [U(f, Q_n) - L(f, Q_n)] + [U(f, R_n) - L(f, R_n)] < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so that $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$, which shows that $\{P_n\}$ is an Archimedean sequence of partitions for f over $[a, b]$. We conclude from the Archimedes-Riemann Theorem that f is integrable on $[a, b]$.