Section 4.1

2. By Proposition 4.4 and Theorem 4.6, $f'(x) = 3x^2 + 2$ for all $x \in \mathbb{R}$. Since $f'(2) = 14$, the desired equation is given by $y = 13 + 14(x - 2)$.

4. **a.** For $x > 0$ with $x \neq 1$,

$$\frac{f(x) - f(1)}{x-1} = \frac{\sqrt{x+1} - \sqrt{2}}{x-1} \cdot \frac{\sqrt{x+1} + \sqrt{2}}{\sqrt{x+1} + \sqrt{2}} = \frac{\cancel{x+1-2}}{(\cancel{x-1})\left(\sqrt{x+1} + \sqrt{2}\right)} = \frac{1}{\sqrt{x+1} + \sqrt{2}}.$$

Now by continuity of the square root function, $1/\left(\sqrt{x+1} + \sqrt{2}\right)$ approaches $1/\left(2\sqrt{2}\right)$ as x approaches 1; we conclude that $f'(1) = 1/\left(2\sqrt{2}\right)$.

b. For $x \neq 1$, $\dfrac{f(x) - f(1)}{x-1} = \dfrac{x^3 + 2x - 3}{x-1} = \dfrac{(\cancel{x-1})(x^2 + x + 3)}{(\cancel{x-1})} = x^2 + x + 3$. Because polynomials are continuous, $x^2 + x + 3$ approaches 5 as x approaches 1; we conclude that $f'(1) = 5$.

c. For $x \neq 1$, $\dfrac{f(x) - f(1)}{x-1} = \dfrac{\frac{1}{1+x^2} - \frac{1}{2}}{x-1} = \dfrac{2 - (1+x^2)}{2(1+x^2)(x-1)} = \dfrac{(\cancel{1-x})(1+x)}{2(1+x^2)(\cancel{x-1})} = \dfrac{-(x+1)}{2(1+x^2)}$.

Since $-(x+1)/\left(2(1+x^2)\right)$ approaches $-1/2$ as x approaches 1; we conclude that $f'(1) = -1/2$.

6. Our assumption that $f : I \to \mathbb{R}$ is differentiable at $h(x_0)$ implies that $\lim_{u \to h(x_0)} \dfrac{f(u) - f(h(x_0))}{u - h(x_0)} = f'(h(x_0))$. Now if $\{x_n\}$ is any sequence in $J \setminus \{x_0\}$ that converges to x_0, then $\{h(x_n)\}$ converges to $h(x_0)$ in $I \setminus \{h(x_0)\}$ by our assumption that h is continuous at x_0. Therefore by definition of limit, $\left\{\dfrac{f(h(x_n)) - f(h(x_0))}{h(x_n) - h(x_0)}\right\}$ converges to $f'(h(x_0))$; the desired conclusion follows.

8. First, $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x)}{x}$ for all $x \neq 0$ because $f(0) = 0$. Next, our assumption implies that $|f(x)| \leq |x|^n$ for all x; because $n \geq 2$, $|x|^n \leq |x|^2$ for all x with $|x| \leq 1$, so that $|f(x)| \leq |x|^2$ for such x. It follows from Exercise 9 of Section 3.7 that $\lim_{x \to 0} \dfrac{f(x)}{x} = 0$, so that f is differentiable at 0. Finally, f is differentiable at all $x \neq 0$ by Proposition 4.4.

10. We show that g is differentiable at $x=1$ if and only if $b=6$ and $a=-3$. First, the quotient $\frac{g(x)-g(1)}{x-1}=\frac{a+bx-3}{x-1}$ for $x>1$. This shows that $\lim_{x\to 1, x>1}\frac{g(x)-g(1)}{x-1}$ does not exist if $b=0$; if $b\neq 0$, then $\frac{g(x)-g(1)}{x-1}=\frac{a+bx-3}{x-1}=\frac{b(x-(3-a)/b)}{x-1}$ for $x>1$, so that $\lim_{x\to 1, x>1}\frac{g(x)-g(1)}{x-1}$ exists if and only if $(3-a)/b=1$, or $a+b=3$, and in this event $\lim_{x\to 1, x>1}\frac{g(x)-g(1)}{x-1}=b$. (These assertions can be verified from the definition of limit using $x_n=1+1/n$.) On the other hand, $\frac{g(x)-g(1)}{x-1}=\frac{3x^2-3}{x-1}=\frac{3(x+1)\cancel{(x-1)}}{\cancel{x-1}}=3(x+1)$ for $x<1$, so that $\lim_{x\to 1, x<1}\frac{g(x)-g(1)}{x-1}=6$. It follows that if g is differentiable if and only if $b=6$ and $a+b=3$, which is to say $b=6$ and $a=-3$.

12. We fix $x_0\in\mathbb{R}$. Our assumption that f is differentiable on $\mathbb{R}$ implies that $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$, and so by definition of limit, if $\{x_n\}$ is any sequence in $\mathbb{R}\setminus\{x_0\}$, then $\left\{\frac{f(x_n)-f(x_0)}{x_n-x_0}\right\}$ converges to $f'(x_0)$. However, by our assumption that f is monotone increasing, $\frac{f(x_n)-f(x_0)}{x_n-x_0}\geq 0$ for all $n\in\mathbb{N}$ (regardless of whether $x_n>x_0$ or $x_n<x_0$); hence by Lemma 2.21 $f'(x_0)\geq 0$.

14. Following the hint, for $h\neq 0$ we have

$$\begin{aligned}\frac{f(x_0+h)-f(x_0-h)}{h}&=\frac{f(x_0+h)-\cancel{f(x_0)}+\cancel{f(x_0)}-f(x_0-h)}{h}\\&=\frac{f(x_0+h)-f(x_0)}{h}+\frac{f(x_0-h)-f(x_0)}{-h}.\end{aligned}$$

Since f is differentiable at x_0, by Exercise 6 each of the last two quotients has a limit of $f'(x_0)$ as $h\to 0$; we take $h(x)=x_0+x$ in the first quotient, and $h(x)=x_0-x$ in the second. Thus the proposed limit exists and equals $2f'(x_0)$.

16. By Exercise 6 with $h(x)=x^2$ we have $\lim_{x\to 0}\frac{f(x^2)-f(0)}{x^2-0}=f'(0)$, that is $\lim_{x\to 0}\frac{f(x^2)-f(0)}{x^2}=f'(0)$. Now for $x\neq 0$ we have $\frac{f(x^2)-f(0)}{x}=\frac{f(x^2)-f(0)}{x^2}\cdot x$, so that $\lim_{x\to 0}\frac{f(x^2)-f(0)}{x}=\lim_{x\to 0}\frac{f(x^2)-f(0)}{x^2}\cdot\lim_{x\to 0}x=f'(x_0)\cdot 0=0$ by Theorem 3.37.

18. First, the definition of f gives $f(0)=1+4\cdot 0+0^2\cdot h(0)=1$. Next, for all $x\in\mathbb{R}\setminus\{0\}$,
$\dfrac{f(x)-f(0)}{x-0}=\dfrac{\cancel{1}+4x+x^2h(x)-\cancel{1}}{x}=\dfrac{4x+x^2h(x)}{x}=4+xh(x)$. Now because h is bounded there exists a number M with $|h(x)|\le M$ for all $x\in\mathbb{R}$. This means that $|xh(x)|\le M|x|$, or $-M|x|\le xh(x)\le M|x|$, for all $x\in\mathbb{R}$; it follows that $\lim_{x\to 0} xh(x)=0$, by reasoning similar to that used in Exercise 9 of Section 3.7. Therefore $\lim_{x\to 0} 4+xh(x)=4+0=4$, which means that $f'(0)=4$.