Section 4.2

2. First, by Proposition 4.13 and Theorem 4.6, f is a composition of differentiable functions, and so is itself differentiable by the Chain Rule; moreover, Proposition 4.15 gives $f'(x) = -x/(1+x^2)^{3/2}$ on $x > 0$. Next, f is strictly decreasing with $f(2) = \sqrt{1/5}$, and so by Theorem 4.11 with $x_0 = 2$, f^{-1} is differentiable at $y_0 = \sqrt{1/5} = f(2)$ with $(f^{-1})'(\sqrt{1/5}) = 1/f'(2) = -5^{3/2}/2$.

4. First, if $x_1, x_2 \in (0,1)$ satisfy $x_1 < x_2$, then $1 + x_1 < 1 + x_2$, so that $1/(1+x_1) > 1/(1+x_2)$, that is, $f(x_1) > f(x_2)$; it follows that f is strictly decreasing. Moreover, f is differentiable by Theorem 4.6 and the Chain Rule, with $f'(x) = -(1+x)^{-2}$ for all $x \in (0,1)$. Next, the inequality $0 < x < 1$ is equivalent to $1 < x + 1 < 2$, that is $1 > f(x) > 1/2$, so that $f(I) \subset J$. Conversely, if $y \in J$, then $1/2 < y < 1$, which gives $2 > 1/y > 1$ and then $1 > 1/y - 1 > 0$; thus $1/y - 1 \in I$ and in fact $f(1/y - 1) = 1/(1/y) = y$. It follows that $f(I) = J$ and that $f^{-1}(y) = 1/y - 1 = (1-y)/y$ for all y in J. Finally, $f^{-1}(y) = \dfrac{y(-1) - (1-y)}{y^2} = \dfrac{-1}{y^2}$ by Theorem 4.6, and formula (4.6) gives $(f^{-1})'(y) = 1/f'(1/y - 1) = -(1 + 1/y - 1)^2 = -1/y^2$ as well.

6. Let $x_0 > 0$. For $x > 0$ with $x \neq x_0$ we have

$$\frac{g(x) - g(x_0)}{x - x_0} = \frac{f(cx) - f(cx_0)}{x - x_0} = c\frac{f(cx) - f(cx_0)}{cx - cx_0} = c\frac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)},$$

where $h : (0, \infty) \to \mathbb{R}$ is defined by $h(x) = cx$. It follows from Exercise 6 of Section 4.1 that $\lim_{x \to x_0} \dfrac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)} = f'(h(x_0)) = f'(cx_0)$, so that $\lim_{x \to x_0} \dfrac{g(x) - g(x_0)}{x - x_0} = cf'(cx_0)$. It follows that g is differentiable with $g'(x) = cf'(cx)$ for all $x > 0$.

8. First, by our assumptions on $\{x_n\}$, the Monotone Convergence Theorem implies that $\{x_n\}$ converges to $x_0 = \sup\{x_n \mid n \text{ in } \mathbb{N}\}$. Next, f is continuous on $\mathbb{R}$ because it is differentiable there, so that $f(x_0) = \lim_{n \to \infty} f(x_n)$. Moreover $f(x_0) \geq f(x_n)$ for all indices n; indeed, if instead $f(x_m) > f(x_0)$ for some $m \in \mathbb{N}$, then our assumption that $\{f(x_n)\}$ is monotone increasing implies that $f(x_n) \geq f(x_0) + \varepsilon$ for all $n \geq m$, where $\varepsilon = f(x_m) - f(x_0) > 0$, and this contradicts the convergence of $\{f(x_n)\}$ to $f(x_0)$. Finally, the facts that f is differentiable at x_0 and that $\{x_n\}$ is strictly increasing, together with the definition of limit, imply that $\left\{\dfrac{f(x_n) - f(x_0)}{x_n - x_0}\right\}$ converges to $f'(x_0)$. But $\dfrac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0$ for all indices n by the above, which by Lemma 2.21 means that $f'(x_0) \geq 0$.