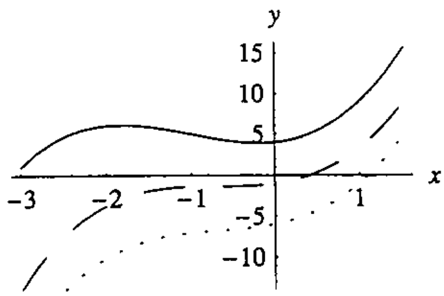
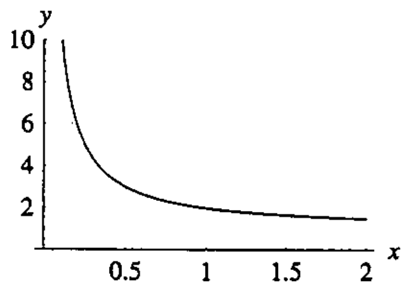
Section 4.3

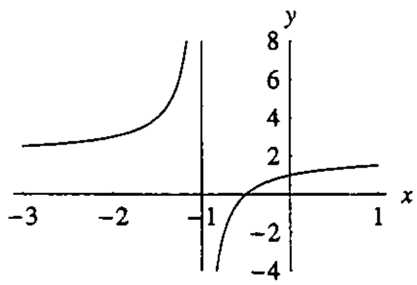
2. **a.** The quadratic formula implies that $f'(x) = 3x^2 + 2ax + b = 0$ for $x = \left(-a \pm \sqrt{a^2 - 3b}\right)/3$ provided that $a^2 \geq 3b$. We consider three cases separately. First, if $a^2 > 3b$, then taking $x = -a/3$ and applying the Intermediate Value Theorem gives $f'(x) < 0$ for $\left(-a - \sqrt{a^2 - 3b}\right)/3 < x < \left(-a + \sqrt{a^2 - 3b}\right)/3$; likewise, since $f'\left(\left(-a \pm 2\sqrt{a^2 - 3b}\right)/3\right) = a^2 - 3b > 0$, $f'(x) > 0$ for $x < \left(-a - \sqrt{a^2 - 3b}\right)/3$ or $x > \left(-a + \sqrt{a^2 - 3b}\right)/3$. (The case $a = 3$, $b = 1$ and $c = 4$ is the solid curve at right.) Next, if $a^2 = 3b$, then the fact that $f'(-a/3 + 1) = -a^2/3 + b + 3$ shows that $f'(x) \geq 0$ for all x. (The case $a = 2$, $b = 4/3$ and $c = -1$ is shown dashed.) Finally, if $a^2 < 3b$, then b is positive, and so $f'(0) = b$ implies that $f'(x) > 0$ for all x. (The case $a = b = 2$ and $c = -6$ is dotted.)



b. Here $h'(x) = -b/x^2 < 0$ for all x, from which it follows that h is decreasing on $(0, \infty)$ for all choices of a and b. The case $a = b = 1$ is shown at right.



4. We suppose to the contrary that the given equation has two solutions $x_1, x_2 \in (0,1)$; we may and do assume that $x_1 < x_2$. Because the function $f : [x_1, x_2] \to \mathbb{R}$ defined by $f(x) = x^3 - 3x + c$ is differentiable on (x_1, x_2), Rolle's Theorem implies that $f'(x_0) = 0$ for some $x_0 \in (x_1, x_2)$; since $f'(x) = 3x^2 - 3$, this means that $3x_0^2 - 3 = 0$. However, this contradicts the fact that $x_0 \in (0,1)$; we conclude that the given equation does not have two solutions.



6. Since $f(x) = x^4 + 2x^2 - 6x + 2$ is continuous on $\mathbb{R}$ with $f(0) = 2$, $f(1) = -1$, and $f(2) = 14$, the Intermediate Value Theorem applied to f over each of the intervals $[0,1]$ and $[1,2]$ shows that the given equation has a solution in each of the disjoint intervals $(0,1)$ and $(1,2)$. Thus the equation has at least two solutions. On the other hand, if the equation has three distinct solutions, then by Rolle's Theorem $f'(x)$ has two distinct roots, and a second application of Rolle's Theorem shows that $f''(x) = 0$ for some $x \in \mathbb{R}$. But $f''(x) = 12x^2 + 4 > 0$ for all x; the contradiction shows that the equation cannot have three solutions, and so has exactly two.

8. First we suppose that the equation $f(x)=0$ has three solutions (in ascending order) x_1, x_2, and x_3, where $f(x)=x^3+ax+b$. As before, by Rolle's Theorem this implies that $f'(x)=0$ for two distinct values of x; since $f'(x)=3x^2+a$, we conclude that $a<0$ and that the values of x in question are $x=\pm\sqrt{-a/3}$, so that $x_1<-\sqrt{-a/3}<x_2<\sqrt{-a/3}<x_3$. Moreover, by Exercise 2a f is decreasing on $\left(-\sqrt{-a/3},\sqrt{-a/3}\right)$; from this and $f(x_2)=0$ it follows that $f\left(\sqrt{-a/3}\right)=-\left(2\sqrt{3}/9\right)(-a)^{3/2}+b<0$ and $f\left(-\sqrt{-a/3}\right)=\left(2\sqrt{3}/9\right)(-a)^{3/2}+b>0$. These inequalities together imply that $-(-a)^{3/2}<9b/2\sqrt{3}<(-a)^{3/2}$, which is equivalent to the desired inequality. Conversely, if $4a^3+27b^2<0$, then again $a<0$ and reversing the above steps gives $f\left(\sqrt{-a/3}\right)<0$ and $f\left(-\sqrt{-a/3}\right)>0$. Moreover, computation shows that $f\left(2\sqrt{-a/3}\right)=f\left(-\sqrt{-a/3}\right)<0$; thus by the Intermediate Value Theorem f has three zeros.

10. First, F would be continuous because it is differentiable. Next, F would be constant on $(-\infty,0)$ by Lemma 4.19. Also, since the function x would have derivative identical to that of F on $(0,\infty)$, by the Identity Criterion $F(x)=A+x$ for some constant A. However, in this event, for $x\neq 0$ the quotient $(f(x)-f(0))/(x-0)$ would be 0 if $x<0$ and 1 if $x>0$, so that $f'(0)$ would not exist, contrary to our assumption that f is differentiable. We conclude that no such function F exists.

12. For a natural number n we let $S(n)$ denote the assertion that for a polynomial p of degree n, the equation $p(x)=0$ has at most n solutions. First, since a polynomial of degree 1 is of the form $p(x)=ax+b$ with $a\neq 0$, and since the equation $ax+b=0$ has exactly one solution, $S(1)$ is true. Next we suppose that $S(k)$ is true for some natural number k, and we choose a polynomial p of degree $k+1$. Now p' is a polynomial of degree k, and so the equation $p'(x)=0$ has at most k solutions; it follows from the preceding exercise that the equation $p(x)=0$ has at most $k+1$ solutions, which proves that $S(k+1)$. Hence by the Principle of Mathematical Induction $S(n)$ is true for all natural numbers n.

14. The function $G:(-1,1)\to\mathbb{R}$ defined by $G(x)=-\sqrt{1-x^2}$ satisfies $G'(x)=x/\sqrt{1-x^2}=g'(x)$ for all $x\in(-1,1)$ by the Chain Rule, and so by the Identity Criterion $g(x)=-\sqrt{1-x^2}+C$ for some constant C and all $x\in(-1,1)$. The condition that $g(0)=25$ then implies that $C=26$, so that $g(x)=-\sqrt{1-x^2}+26$.

16. We define h as in the hint and note by Theorem 4.6 and our assumption that h is differentiable with $h'(x)=2f(x)f'(x)+2g(x)g'(x)=0$ for all x; it follows from Lemma 4.19 that there exists a constant c such that $h(x)=c$, that is $[f(x)]^2+[g(x)]^2=c$, for all $x\in\mathbb{R}$. In light of our assumption, taking $x=0$ shows that $c=1$, so that $[f(x)]^2+[g(x)]^2=1$.

18. Our assumption that g is differentiable on I implies that it is continuous there, and therefore h is continuous on $I \setminus \{x_0\}$. On the other hand, our assumption that g is differentiable implies that $\lim_{x\to x_0} h(x) = g'(x_0) = h(x_0)$ by definition of derivative, so that by Exercise 8c of Section 3.7, h is continuous at x_0 as well; it follows that h is continuous on I. The second assertion in the exercise follows immediately from the preceding exercise.

20. If $x \neq 0$, then $\dfrac{f(x)-f(0)}{x-0} = \dfrac{x \pm x^2 - 0}{x} = 1 \pm x$; thus, if $\{x_n\}$ is any sequence in $\mathbb{R} \setminus \{x_0\}$ that converges to 0, then the sequence $\left\{\dfrac{f(x_n)-f(0)}{x_n - 0}\right\}$ converges to 1. It follows that $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = 1$, that is, that $f'(0) = 1$. However, given any $\delta > 0$, by the density of the rational in $\mathbb{R}$ we can choose a rational number x in $(0,\delta)$. Then by the density of the irrationals in $\mathbb{R}$ we can find a sequence $\{x_n\}$ of irrational numbers in $(0,x)$ that converges to x. Now for each index n we have $f(x_n) - f(x) = x_n + x_n^2 - (x - x^2) = (x_n - x) + (x_n^2 + x^2)$; by definition of convergence, there exists $N \in \mathbb{N}$ such that $x_n > x - x^2/2$, or $x_n - x > -x^2/2$, if $n \geq N$; thus $f(x_n) - f(x) > -x^2/2 + (x_n^2 + x^2) = x_n^2 - x^2/2$ for such n. Finally, again by definition of convergence we can find $M \in \mathbb{N}$ such that $x_n > x/\sqrt{2}$, or $x_n^2 > x^2/2$, if $n \geq M$; it follows that $f(x_n) - f(x) > x_n^2 - x^2/2 > 0$ if $n \geq \max\{N, M\}$. Since $x_n < x$ but $f(x_n) > f(x)$ for such n, we conclude that f is not monotone increasing on $(0,\delta)$.

22. First, if $x > 0$, then by the Mean Value Theorem applied to $f:[0,x] \to \mathbb{R}$, we can find $x_0 \in (0,x)$ with $f(x) - f(0) = f'(x_0)(x-0)$; it follows from our assumption that $f(x) - f(0) \geq cx$, or $f(x) \geq f(0) + cx$. A similar argument shows that $f(x) \leq f(0) + cx$ if $x < 0$. Now if y is any real number, then let $x = (y - f(0))/c$; if $x \geq 0$, then the above inequalities imply that $f(x) \geq f(0) + cx = y$, and similarly if $x \leq 0$, then $f(x) \leq f(0) + cx = y$. Moreover f is continuous because it is differentiable; therefore by the Intermediate Value Theorem we can find $x_0 \in \mathbb{R}$ with $f(x_0) = y$, so that $f(\mathbb{R}) = \mathbb{R}$.

24. Not necessarily; as a counterexample we may take $f(x) = \begin{cases} x^n & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$, where n denotes an even natural number greater than 2. First, $f(0) = 0$. Next, for $x \neq 0$ the difference quotient $\dfrac{f(x)-f(0)}{x-0}$ is x^{n-1} if $x < 0$ and 0 if $x > 0$; since $n-1$ is a natural number, it follows from the definition of derivative that $f'(0) = 0$. Thus $f'(x) = \begin{cases} nx^{n-1} & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$, and because $n-1$ is odd, $f'(x) \leq f(x)$ for all x. Finally, for $x \neq 0$ the difference quotient $\dfrac{f'(x)-f'(0)}{x-0}$ is nx^{n-2} if $x < 0$ and 0 if $x > 0$; since $n-2$ is a natural number, the definition of derivative implies that $f''(0) = 0$. Thus $f''(x) = \begin{cases} n(n-1)x^{n-2} & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$, so that f has two derivatives.