Section 4.4

2. If $x \in \mathbb{R}$, then Theorem 4.24 with $I = \mathbb{R}$ and $n = 6$ implies that there is a point z strictly between x_0 and x with $p(x) = \frac{p^{(6)}(z)}{6!}(x - x_0)^6$. But our assumption on p implies that $p^{(6)} \equiv 0$, so that $p(x) = 0$.

4. Let $u, v \in [a,b]$. If $u = v$, then both sides of the proposed inequality are 0, and so the inequality holds. Otherwise we may assume without loss of generality that $u < v$. In this case the Cauchy Mean Value Theorem applies to f and g over the interval $[u,v]$ to show that there exists $x_0 \in (u,v)$ with $\frac{f(u) - f(v)}{g(u) - g(v)} = \frac{f'(x_0)}{g'(x_0)}$, or $f(u) - f(v) = \frac{f'(x_0)}{g'(x_0)}[g(u) - g(v)]$. Thus $|f(u) - f(v)| = \frac{|f'(x_0)|}{|g'(x_0)|}|g(u) - g(v)|$. Our assumption that $|f'(x)| \geq |g'(x)| > 0$ for all $x \in (a,b)$ implies that $\frac{|f'(x_0)|}{|g'(x_0)|} \geq 1$, which gives the result.

6. We let $S(n)$ denote the assertion that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ whenever $f:(-1,1) \to \mathbb{R}$ satisfies the given assumptions. First, if f is differentiable and there is a positive number M such that $|f(x)| \leq M|x|$ for all $x \in (-1,1)$, then choosing $x = 0$ implies that $|f(0)| \leq M|0| = 0$, so that $f(0) = 0$, proving $S(1)$. Next, we choose a natural number k and suppose that $S(k)$ is true. If $f:(-1,1) \to \mathbb{R}$ has $k+1$ derivatives and there is a positive number M such that $|f(x)| \leq M|x|^{k+1}$ for all $x \in (-1,1)$, then $|f(x)| \leq M|x|^k$ for all $x \in (-1,1)$ as well, since $|x|^{k+1} \leq |x|^k$ for such x. The truth of $S(k)$ then implies that $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$. But now by Theorem 4.24, for each $x \in (-1,1) \setminus \{0\}$ there exists $z(x)$ strictly between x and 0 with $f(x) = \frac{f^{(k)}[z(x)]}{k!}x^k$, or $f^{(k)}[z(x)] = \frac{k!f(x)}{x^k}$; thus our assumption on f gives

$$\left|f^{(k)}[z(x)]\right| = k!\frac{|f(x)|}{|x|^k} \leq k!\frac{M|x|^{k+1}}{|x|^k} = M \cdot k! \cdot |x|.$$

Therefore, if $\{x_n\}$ is any sequence in $(-1,1) \setminus \{0\}$ that converges to 0, the sequence $\{f^{(k)}[z(x_n)]\}$ converges to 0 also, by the Comparison Lemma, which means that $\lim_{x \to 0} f^{(k)}[z(x)] = 0$. Moreover, the assumption that f has $k+1$ derivatives implies that $f^{(k)}$ is differentiable, and hence continuous, on $(-1,1)$; altogether, then, $0 = \lim_{x \to 0} f^{(k)}[z(x)] = f^{(k)}(0)$. Therefore $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = f^{(k)}(0) = 0$, proving $S(k+1)$. It follows from the Principle of Mathematical Induction that $S(n)$ is true for all natural numbers n.

8. We let $S(n)$ denote the assertion that Leibnitz's formula is true whenever f and g have n derivatives. Then $S(1)$ states that

$(fg)'(x)=\sum_{k=0}^{1}\binom{1}{k}f^{(k)}(x)g^{(1-k)}(x)=f(x)g'(x)+f'(x)g(x)$, which is true by the Product Rule. Next we assume that $S(m)$ is true for some natural number m, and we assume that $f:I\to\mathbb{R}$ and $g:I\to\mathbb{R}$ have $m+1$ derivatives. The truth of $S(m)$ gives

$(fg)^{(m)}(x)=\sum_{k=0}^{m}\binom{m}{k}f^{(k)}(x)g^{(m-k)}(x)$, and then by Theorem 4.6(ii) (the "product rule"),

$$\begin{aligned}(fg)^{(m+1)}(x)&=\left[\sum_{k=0}^{m}\binom{m}{k}f^{(k)}(x)g^{(m-k)}(x)\right]'=\sum_{k=0}^{m}\binom{m}{k}\left[f^{(k)}(x)g^{(m+1-k)}(x)+f^{(k+1)}(x)g^{(m-k)}(x)\right]\\&=\sum_{k=0}^{m}\binom{m}{k}f^{(k)}(x)g^{(m+1-k)}(x)+\underline{\sum_{k=0}^{m}\binom{m}{k}f^{(k+1)}(x)g^{(m-k)}(x)}\\&=\sum_{k=0}^{m}\binom{m}{k}f^{(k)}(x)g^{(m+1-k)}(x)+\sum_{k=1}^{m+1}\binom{m}{k-1}f^{(k)}(x)g^{(m+1-k)}(x);\end{aligned}$$

the third line results from replacing the index k in the underlined summation with $k-1$. Now it was proved in Exercise 21 of Section 1.3 that $\binom{m}{k}+\binom{m}{k-1}=\binom{m+1}{k}$ for all natural numbers k with $k\le m$. Thus, isolating the terms corresponding to $k=0$ and $k=m+1$ shows that $(fg)^{(m+1)}(x)$ is

$$\begin{aligned}&f(x)g^{(m+1)}(x)+\sum_{k=1}^{m}\binom{m}{k}f^{(k)}(x)g^{(m+1-k)}(x)+\sum_{k=1}^{m}\binom{m}{k-1}f^{(k)}(x)g^{(m+1-k)}(x)+f^{(m+1)}(x)g(x)\\&=f(x)g^{(m+1)}(x)+\sum_{k=1}^{m}\binom{m+1}{k}f^{(k)}(x)g^{(m+1-k)}(x)+f^{(m+1)}(x)g(x)=\sum_{k=0}^{m+1}\binom{m+1}{k}f^{(k)}(x)g^{(m+1-k)}(x),\end{aligned}$$

which proves $S(m+1)$. We conclude from the Principle of Mathematical Induction that $S(n)$ is true for all natural numbers n.