Mortality Models and Longevity Risk for Small Populations

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Abstract

Prolonging life expectancy and improving mortality rates is a common trend of the 21st century. Stochastic models, such as Lee-Carter model (Lee and Carter, 1992), are a popular choice to deal with longevity risk. However, these mortality models often have unsatisfactory results for the case of small populations. Thus, quite a few modifications (such as approximation and maximal likelihood estimation) to the Lee-Carter can be used for the case of small populations or missing observations. In this study, we propose an alternative approach (graduation methods) to improve the performance of stochastic models.

The proposed approach is a combination of data aggregation and mortality graduation. In specific, we first combine the historical data of target population, treating it as the reference population, and use the data graduation methods (Whittaker and partial standard mortality ratio) to stabilize the mortality estimates of the target population. We first evaluate whether the proposed method have smaller errors in mortality estimation than the Lee-Carter model in the case of small populations, and explore if it is possible to reduce the bias of parameter estimates in the Lee-Carter model. We found that the proposed approach can improve the model fit of the Lee-Carter model when the population size is 200,000 or less.

Keywords: Longevity Risk, Small Area Estimation, Lee-Carter Model, Standard Mortality Ratio, Graduation

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1. Introduction

Since people are living longer, life planning for the elderly becomes a popular issue around the world. The study of the elderly's mortality rates and health receives a lot of attention among all elderly research topics. However, the human life expectancy has been increasing rapidly and many countries (especially for those with small populations and with a rapid increase in longevity) do not have enough historical data, including the size and the period, which makes modeling the mortality rates difficult. For example, the population records of highest attained age in Taiwan are 100 years old in 1998 (95 years old in 1995), i.e., limited data points for modeling the elderly mortality rates.

Dealing with estimating mortality rates of small populations is not new in insurance industry and the actuaries often apply smoothing methods to reduce the fluctuations of age-specific mortality rates in constructing life tables. Increasing the sample size is a natural choice to stabilize the estimates. However, it is often not possible to collect more data and thus the graduation methods are used instead. Traditional graduation methods, such as moving weighted average (London, 1985), can be treated as augmenting mortality data from neighboring ages. Now the graduation is a common (and often necessary) process for constructing life tables and estimating mortality rates. The Human Mortality Database (Source: University of California, Berkeley) also makes adjustment in the case where the data size is small. For the group of ages 80 and over, mortality rates are adjusted by the method of extinct generations, while the survivor ratio method is used over age 90 (Thatcher et al., 2002).

The need for increasing the sample size (or graduation) also appears in improving the parameter estimation of mortality models. For example, Lee-Carter model (Lee and Carter, 1992) is a popular mortality model among governments and insurance companies. However, applying the Lee-Carter to small populations (such as Nordic countries and Denmark) would produce estimation bias in age-related parameters (Booth et al., 2006; Jarner and Kryger, 2011). Similar problem also appears in the CBD model (Cairns et al., 2006) and Chen et al. (2017) found that small population size has a significant influence on the variation of parameters' estimates. Referencing the mortality profile of larger populations is a popular choice for stabling the parameter estimation. For example, the coherent Lee-Carter model by Li and Lee (2005) can reduce the estimation errors by referencing the mortality data from populations with similar mortality improvements.

This probably is the main reason why many studies focus on modeling mortality rates for small populations recently. Intuitively, increasing the sample size is the most efficient way to stabilize the parameter estimation of mortality models, and including the mortality data from neighboring areas (or areas with similar mortality profiles) is a natural choice. We think that it is possible to adapt the idea of graduation methods and use it to stabilize the parameter estimation of mortality models in the case of small populations. In this study, we explore the possibility of combining graduation methods and coherent Lee-Carter model, adjusting the differences between the small and the reference populations. Wang et al. (2012) showed that the Whittaker and partial standard mortality ratio (SMR) methods are effective in reducing bias and variation of mortality estimates if a proper reference population is chosen. We shall further evaluate if the idea of Wang et al. can be used to stabilize the parameter estimation of mortality models.

We propose a modification in estimating the parameters of Lee-Carter model for small populations. Similar to the setting in Wang et al., under certain mortality scenario, we use computer simulation to evaluate whether the modification via graduation is valid with respect to MAPE (Mean Absolute Percentage Error). Also, we use the empirical data from Taiwan and Taipei city to check the proposed method. The results show that the Whittaker and partial SMR methods can improve the parameter estimation of Lee-Carter model for small populations.

2. Methodology

Increasing the population size (and its similar form) is probably the universal way to deal with the problem associated with small population size, and the extra samples can be from the same population or different populations. Choosing a larger reference population, i.e. a different population, is a popular approach in recent years. For example, there are quite a few modifications of Lee-Carter model, such as the coherent model by Li and Lee (2005) and the three-way decomposition by Russolillo et al. (2011). Modifications of other mortality models are also possible. For example, for the age-period-cohort model, Jarner and Kryger (2011) modified the frailty model into the SAINT model, Cairns et al. (2011) considered a Bayesian approach, and Dowd et al. (2011) proposed a gravity model. Madrigal et al (2011) suggested adding mortality related variables (or extra predictors) in the generalized linear models for modelling life expectancy.

However, it is required that the small and reference populations share similar mortality profile to guarantee whether increasing the population size can improve the parameter estimation. In this study, we first evaluate the effect of similarity level on the model fit and then propose a graduation-based method to increase the population size based on the historical data of small population. Our proposed method can reduce the bias in the parameter estimation, if the small population does not show sudden changes in mortality rates. We should first give a brief introduction to the graduation methods used in this section.

We will first introduce the proposed (partial SMR) approach and evaluate its performance in the next section. The partial SMR (Lee, 2003) is a modification of SMR (standard mortality ratio), which is used to smooth mortality rates of small populations via the information from a large population with respect to SMR. The SMR is often used in epidemiology and is defined as

$$\text{SMR} = \frac{\sum_x d_x}{\sum_x e_x} = \frac{\sum_x d_x}{\sum_x P_x \times m_x^R} \tag{1}$$

where d_x and e_x are the observed and expected numbers of deaths at age x for the small population, P_x is the population size of age x for the small population, and m_x^R is the central death (or mortality) rate of age x for the reference (or large) population. In general, SMR greater/less than 1 usually indicates the small population has a higher/lower overall mortality rate than the reference population, i.e., the SMR can be treated as a mortality index.

If certain age groups in the small population are not many, the observed mortality rate would fluctuate considerably. The SMR can be used, therefore, to refine the mortality rate. For the partial SMR, the graduated mortality rates satisfy

$$v_x = u_x^* \times \exp\left(\frac{d_x \times \hat{h}^2 \times \log(d_x / e_x) + (1 - d_x / \sum d_x) \times \log(\text{SMR})}{d_x \times \hat{h}^2 + (1 - d_x / \sum d_x)} \right), \tag{2}$$

or the weighted average between raw mortality rates and SMR, where $\hat{h}^2$ is the estimate of parameter h^2 for measuring the heterogeneity (in mortality rates) between the small area and large area. u_x^* is the mortality rate for age x in the large population.

The idea behind the partial SMR is similar to credibility weighted estimate for calculating future premium (Klugman et al., 2012), where the estimate is a linear combination of recent observed loss and related reference information. The Bayesian graduation methods (e.g., Kimeldorf and Jones, 1967) function in a similar format and the updated (or posterior) estimate is also a linear combination of new observation and past (or prior) experience (London, 1985). The key is to choose appropriate weight and related reference population.

To achieve satisfactory results, Lee (2003) suggests the weight of partial SMR:

$$\hat{h}^2 = \max\left(\frac{\sum\left((d_x - e_x \times SMR)^2 - \sum d_x\right)}{SMR^2 \times \sum e_x^2} \text{ , } 0 \right) \tag{3}$$

The larger $\hat{h}^2$ is, the larger the difference in age-specific mortality rates (i.e., mortality heterogeneity, or larger dissimilarity in shape between the age-specific mortality curve of the small population and that of the larger population). When the number of deaths is smaller, there will be greater weight from the large population, and the graduated mortality value equals $\mathrm{SMR} \times u_x^*$ when the number of deaths is zero. Lee (2003) mentioned that using the weight function $\hat{h}^2$ in equation (3) usually has smaller mean square error (MSE) in mortality estimation. However, the derivation of $\hat{h}^2$ is through some sort of approximations and it cannot guarantee to have the smallest MSE.

For the reference population, there are quite a few possibilities. A natural choice is the whole nation (or nearby areas) if the small population is a subset of the nation. However, the differences in mortality within a country can be huge, even for neighboring areas. For example, the 2009-2011 life expectancies of men and women in Taipei city (Taiwan's capital) are 79.35 and 84.31, respectively. These numbers are significantly larger than those in eastern Taiwan (70.07 and 78.43) which is about 100 miles from Taipei city. Another possibility is to construct mortality index and to select populations having similar mortality profile as the small population. However, like SMR, populations with same/similar mortality index can have significantly different life expectancy.

In this study, we suggest aggregating the historical data of small population and treat the aggregation as the reference population. Since of the mortality improvement, the life expectancy and mortality rates of the aggregation data are definitely different to those of small population, and their differences are larger if more years of data are involved. Still, as long as there are no rapid changes in mortality improvement, aggregation data are likely to be a good candidate of reference population. The number of years chosen should match to the size of small population. Our suggestion is to collect at least one million. In the next two sections, we will show that the partial SMR can also be used to forecast mortality rates for small populations.

We also consider another graduation methods, Whittaker and Whittaker ratio. The Whittaker graduation method (London, 1985) is to minimize

$$\mathrm{M} = \mathrm{F} + hS = \sum_{x=1}^{\mathrm{n}} w_x (v_x - u_x)^2 + h \sum_{\mathrm{x}=1}^{\mathrm{n\text{-}z}} (\Delta^z v_x)^2 \,, \tag{4}$$

where u_x and v_x are observed and graduated mortality rate/ratio for age *x*, respectively; w_x is weighted number for age *x*; *n* is the number of ages (or age-groups) considered; and *h* and *z* are the parameters to be decided. In this study, we let *h* be the average population size of a single age/5-age group and $z = 3$, as suggested in our previous studies. The Whittaker ratio is an extended version and we plug into the mortality ratio of small and reference populations for graduation, instead of mortality rates.

3. Small Area Estimation

We use the computer simulation to evaluate the performance of partial SMR method in this section and compare it with the Lee-Carter model in the next section. The datasets used in this study are the population of Taiwanese male and both the formats of single age and 5-age group are considered. The population size of Taiwanese male is about 11.5 millions. Also, for the study period, the proportion of elderly was between 6% and 11% and the life expectancy at age 0 was between 71 and 76 years.

We should first show the parameter estimation of Lee-Carter model in the case of small populations. In specific, we use Taiwan’s data to fit the Lee-Carter model and treat the estimated parameters as the true values. We use the mortality data in 1990-2009 to derive the parameters of Lee-Carter model, with the age range 0-84. And then we generate numbers of deaths based on Poisson assumption, producing simulated age-specific mortality rates under different population sizes. The population sizes considered in the computer simulation are from 10,000, 20,000, …, 2 million, and 5 million.

Figures 1 and 2 are the averages and standard errors of Lee-Carter model parameters α_x and β_x from 1,000 simulation runs, under different population sizes. Apparently, the small population size does cause problems in estimation and, in particular, it seems that the β_x estimate is more sensitive. The bias is obvious (or noticeable) when the population size is not more than 50,000 (or 500,000). Applying the idea of t-ratio, or average/standard error, to the parameter estimates, we can further evaluate the estimation of Lee-Carter model. Heuristically, if the t-ratio value of 1.96 is treated as the threshold of unacceptable estimation, it seems that the Lee-Cater model should be applied with care when the population size is not more than 50,000 (Figure 3). In fact, to be more conservative, we think that at least one million people are suggested and any population sizes smaller need some modifications.

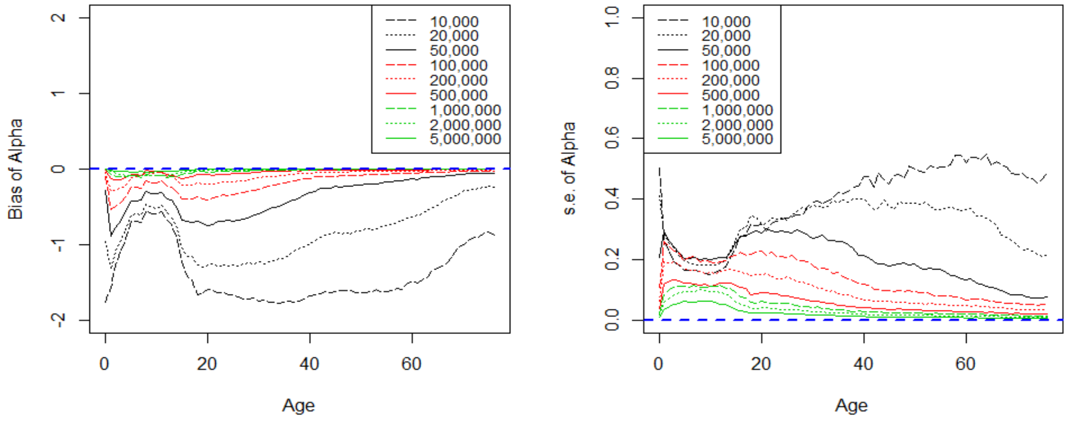

Figure 1. Averages and Standard Errors of α_x from Lee-Carter Model

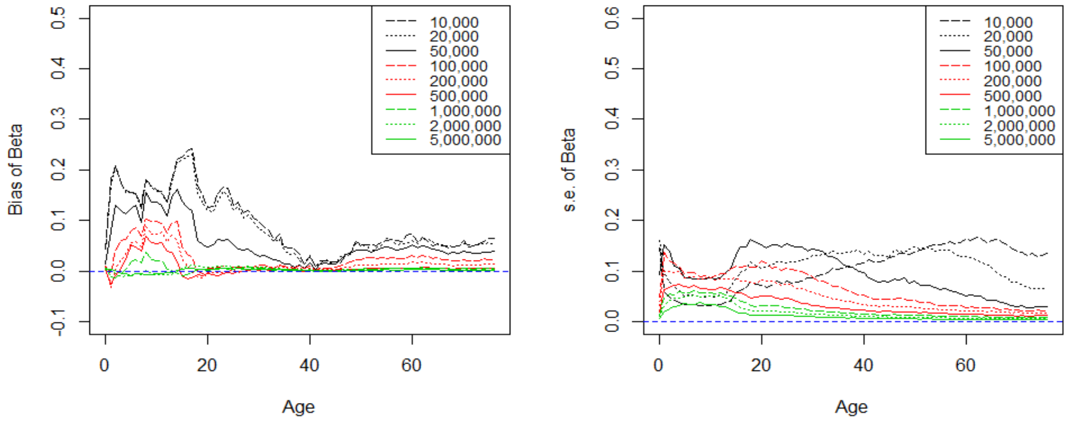

Figure 2. Averages and Standard Errors of β_x from Lee-Carter Model

The preceding simulation results of Lee-Carter model are from the singular value decomposition (SVD). Of course, other estimation methods of Lee-Carter model, including the approximation method, the modifications via maximum likelihood estimation and weighted least squares (Wilmoth, 1993), and Poisson approach (Brouhns et al., 2002) can be used as well. However, the estimation methods do not have significant impacts on the results, although the estimates via SVD have slightly larger bias.

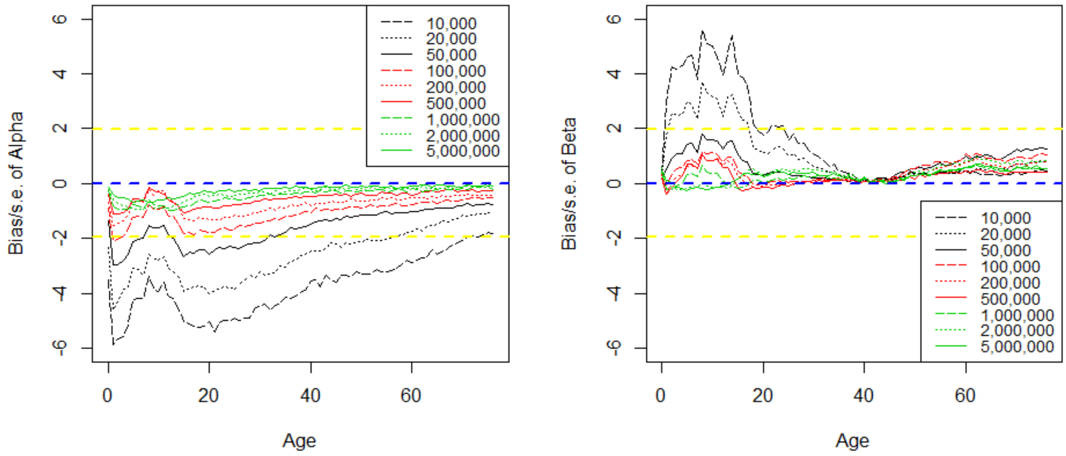

Figure 3. "T-ratio" of $\hat{\alpha}_x$ and $\hat{\beta}_x$ from Lee-Carter Model

We will check if the partial SMR can handle the fluctuation of mortality rates from small populations. In addition to the assumption of population sizes, we also consider various mortality scenarios, connecting the relationship between small population and reference population. Figure 4 shows three mortality scenarios in this study, indicating the mortality ratios (or $s_x = \frac{q_x}{q_x^R}$, where q_x and q_x^R are the mortality rates of age x for the small and reference populations) between the mortality rates of small population and reference population. For the constant scenario, we assume that the shape of mortality rates in small population is similar to that of large population. For the other two scenarios, we assume that the mortality curves of two populations are not identical, e.g., small population has higher mortality rates in the elderly.

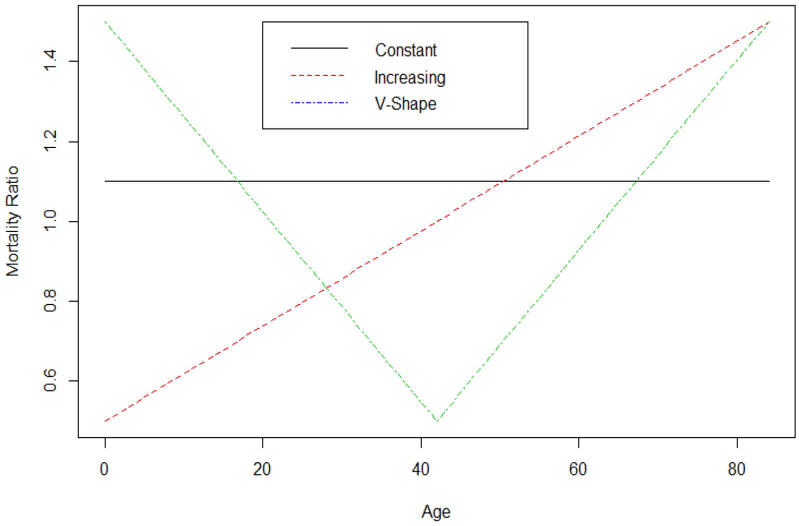

Figure 4. Scenarios of Mortality Ratio

Note that the SMR can be treated as a measure (or similarity index) between the reference population and the small population. The partial SMR is a modified version of the SMR and it is a weighted average of SMR and the observed mortality rates. The partial SMR can adjust the mortality rates of the small population according to observed mortality rates. If the age-specific mortality rates of small population are similar to those of the reference population, i.e., constant scenario, then the SMR serves as a good reference index. However, if the mortality rates of the small population behave differently than those of the reference population, then the SMR would create biased estimates, such as in the case of increasing and V-shape scenarios.

The size of large population equals Taiwan's male population (about 11 million), while the small population has 50,000 and 200,000, with same age structure as the large population. We also apply Whittaker graduation (London, 1985) to the mortality ratio s_x, to compare with the partial SMR. The mechanism behind Whittaker graduation is similar to the moving average, by including the populations at neighboring ages to reduce variance of mortality rates. The reason for considering the Whittaker ratio is to check if the partial SMR "borrow" too much information from reference population when in fact the small population does not look like reference population.

In addition, we add one more parameter a to measure the mortality discrepancy between small and reference populations. For the constant scenario, the mortality ratio of all ages is $s_x = 1 + a$, and the mortality ratio satisfies $1 - a \le s_x \le 1 + a$ for the other scenarios, with $0 \le a \le 0.9$ for all three scenarios. We use MAPE (mean absolute percentage error) of mortality rates to measure the differences between estimated values and true values.

We only show the results of population size 50,000. The simulation results of population size 200,000 are similar and can be found in Appendix A. All simulation results are based on 1,000 replications. Table 1 shows the estimation errors of raw data, Whittaker ratio graduation, and the partial SMR. As expected, in the constant scenario case where the mortality curves of small and reference populations are very similar, the SMR is a good index for smoothing mortality rates of small population and thus the estimation errors (MAPE) of partial SMR are very small. Similar results apply to the Whittaker ratio as well.

In the cases of increasing and V-shape scenarios, the SMR alone is not a good measurement for connecting small and reference populations. The partial SMR has smaller MAPE than the Whittaker ratio in the case of smaller a (or $a \le 0.4$), but the Whittaker ratio is better for larger a when two populations are very different. The larger a is, the larger discrepancy between the mortality rates of small and large populations and it would be dangerous to rely on the large population when the mortality rates of two populations behave differently. It seems that the partial SMR is a feasible approach but we should choose the reference population carefully.

Table 1. MAPE (%) of 3 Mortality Scenarios (Population size = 50,000)

a		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Constant	Raw	30.1	28.7	27.6	26.5	25.3	24.7	23.3	23.0	22.4	21.8
	Whittaker	13.2	12.8	12.5	12.3	11.9	11.7	11.5	11.4	11.1	10.9
	PSMR	5.0	4.6	4.6	4.2	4.1	4.1	3.9	3.8	3.7	3.7
Increasing	Raw	30.2	32.4	36.0	41.3	47.6	56.2	66.2	79.8	99.2	143.3
	Whittaker	13.3	15.1	18.5	23.2	28.9	36.3	45.3	57.9	78.0	122.3
	PSMR	4.9	7.5	12.7	19.4	27.4	37.0	48.8	64.9	88.8	140.9
V-shape	Raw	30.2	31.0	33.6	38.0	43.7	51.4	60.1	72.7	90.9	130.6
	Whittaker	13.3	14.2	17.1	21.5	27.1	33.7	41.4	52.4	68.7	105.8
	PSMR	4.9	7.5	12.5	18.8	26.4	35.2	45.4	59.1	79.2	123.0

4. Modeling Mortality Rates using Graduation Methods

In this section, we use computer simulation to check if the partial SMR can be used to modify the parameter estimation of Lee-Carter model. In specific, we assume that both the mortality rates of small and reference populations satisfy the Lee-Carter model. First, we explore the number of years required for data aggregation using the partial SMR, following by comparing estimates of mortality rates between the Lee-Carter model and the partial SMR. Again, we use the age structure of Taiwanese male population as the underlying population (for both small and reference populations) and apply data from 1990-2009 to obtain parameters (α_x, β_x, and κ_t) of Lee-Carter model, treating them as the true values. Then we simulate the numbers of deaths from Poisson distribution, following by applying the Lee-Carter model (SVD) or the partial SMR. This process is repeated for 1,000 times.

We consider various setting of data aggregation, as shown in Table 2. We use

10% of MAPE as a threshold, marked as “accurate” by Lewis (1982), to judge if the estimation is stable. On average, aggregating 15 years of data has the smallest MAPE’s, following by aggregating 10 years of data. It seems that more years of data used do not guarantee better estimation. We use the mortality improvement via the parameters β_x and κ_t in the Lee-Carter model to explain why.

Table 2. MAPE (%) of Various Aggregation Years (5-age)

	10,000	20,000	50,000	100,000	200,000	500,000
PSMR+20 Years	14.31	11.75	9.68	8.70	8.09	7.50
PSMR+15 Years	13.47	10.78	8.72	7.78	7.10	6.39
PSMR+10 Years	13.80	11.21	9.19	8.25	7.51	6.57
PSMR+ 5 Years	14.83	12.50	10.67	9.75	8.93	7.73

The product of β_x and κ_t equals to the annual mortality improvement for age x and it is about 2% on average in Taiwan, over 3% (or 4%) for certain ages. Heuristically speaking, if we aggregation 20 years of data (1990-2009) and treat it as the reference population of data in 2009, the difference between the small and reference populations would be at least 20% or more. Based on the information from Tables 1-1~1-3, It is not surprise that aggregating 10 or 15 years of data has smaller estimation errors than 20 years. In practice, we suggest aggregating 10~20 years of data for small populations with at least population 50,000.

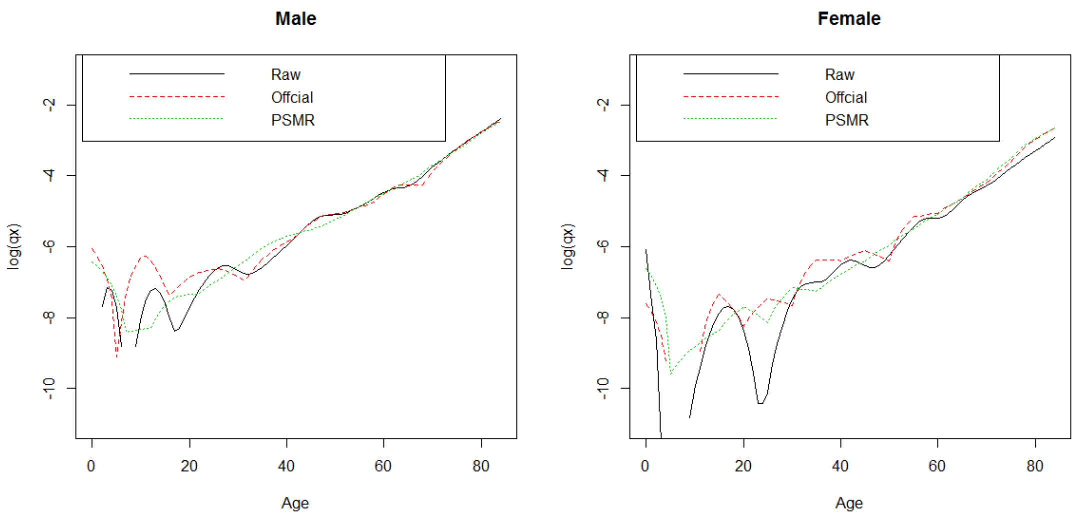

Figure 5. Age-specific Mortality Rates of Pen-Hu County (2014)

We use the Pen-Hu county in Taiwan as a demonstration of partial SMR method. Pen-Hu is an island county, with about 50,000 men and women each. In order to reach one million people, we suggest aggregating at least 20 years of data. Figure 5 shows the mortality rates of 2014 Pen-Hu county from raw data, official abridge life table, and partial SMR (aggregating data of 1995-2014). Both the mortality curves of raw data and abridge life table show dis-continuity around age 10 since there are no observed deaths in 2014. On the other hand, the mortality curves of partial SMR are smoother and show no dis-continuities. It seems that the partial SMR method can be used to graduate mortality rates from small populations.

Table 3. MAPE of Simulation (Taiwanese Male, 1990-2009, 5-age)

(Unit: %)

	10,000	20,000	50,000	100,000	200,000	500,000
Raw	68.23	50.59	32.90	22.88	16.28	10.27
Lee-Carter	33.57	23.67	15.53	10.97	8.66	6.05
Partial SMR	14.31	11.75	9.68	8.70	8.09	7.50

Table 4. MAPE of Simulation (Taiwanese Male, 1990-2009, Single-age)

(Unit: %)

	10,000	20,000	50,000	100,000	200,000	500,000
Raw	125.56	101.45	73.01	54.89	39.40	24.60
Lee-Carter	69.77	52.70	31.11	19.93	13.30	8.32
Partial SMR	18.02	14.82	12.47	11.39	10.72	10.19

For comparing the partial SMR and the Lee-Carter model, we aggregate 20-year data and treat it as the reference population. Tables 3 and 4 are the MAPEs of different methods from 1,000 computer replications, in the format of 5-age and single-age groups. The MAPEs are larger for the case of single-age groups, about two times more. For both the 5-age and single-age groups, the partial SMR has smaller MAPE than the Lee-Carter model when the population size is not more than 200,000. The advantage is more obvious for smaller population size and single-age group. It is interesting that the partial SMR does not suffer much for smaller population size and its MAPE is always smaller than 20%. It seems that the partial SMR can be used to handling mortality estimation of small population.

Other than the MAPE's of mortality rates, we can also use the survival curve to validate the proposed approach. The role of survival curve is similar to life expectancy and thus can link to longevity. The differences between the theoretical survival curve and the predicted survival curves (from Lee-Carter estimation and the proposed approach) can be used to evaluate the impact of estimation methods on the life insurance products. We first use the theoretical survival curves in 1995 and 2005 for population size 10,000, together with their estimates via the Lee-Carter method and proposed approach, to demonstrate the differences of estimation methods (Figure 6, 1,000 simulation runs). The areas between the theoretical survival curves and those of the proposed partial SMR are significantly smaller, which indicates that the proposed approach has a better performance in estimating the life expectancy.

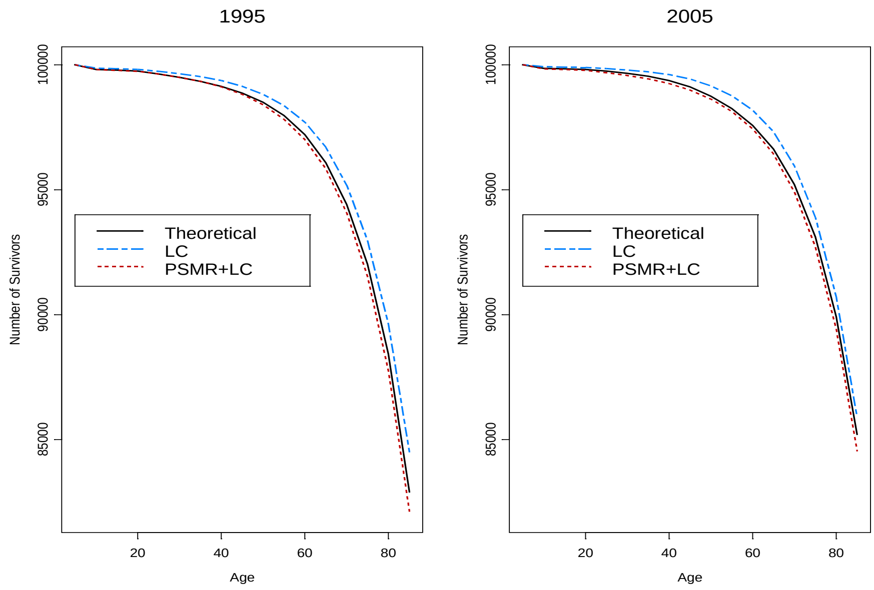

Figure 6. Survival Curves and Their Estimates

The prediction of life expectancy can be done in a similar way and we can use the increment of life expectancy to evaluate the proposed approach. Note that the rectangularization of the survival curve becomes more obvious because the mortality improvement occurs at all ages (Yue, 2012). Thus, the survival curve will move to the right with time and the areas between the survival curves of two different years can be used to measure the level of life prolonging. In particular, we consider three types of setting: 1995 vs. 2005 and 2005 vs. 2015 (ten years apart) and 1995 vs. 2015 (20 years apart). Table 5 shows the bias of areas between two survival curves, comparing to the theoretical results, based on 1,000 simulation runs. Positive/negative values indicate under/over-estimates in the prediction of prolonging life and pricing of life insurance products. On average, the partial SMR has smaller absolute bias and the Lee-Carter model tends to have large negative bias for smaller population sizes, indicating under-estimates in pricing life annuity products.

Table 5. Bias of Survival Curves' Estimates

		10,000	20,000	50,000	100,000	200,000	500,000
1995 vs. 2005	Lee-Carter	-8%	2%	0%	7%	9%	10%
	Partial SMR	-1%	5%	5%	6%	6%	5%
2005 vs. 2015	Lee-Carter	-43%	-30%	-16%	-14%	-17%	-8%
	Partial SMR	-5%	-7%	-1%	-4%	-6%	-5%
1995 vs. 2015	Lee-Carter	24%	-12%	-7%	-2%	-3%	2%
	Partial SMR	-3%	-1%	2%	2%	1%	1%

Table 6. MSE of Parameters' Estimates (Taiwanese Male, 1990-2009, 5-age)

(Unit: 10^{-3} for α_x & 10^{-4} for β_x)

		10,000	20,000	50,000	100,000	200,000	500,000
α_x	Lee-Carter	169.02	47.19	10.55	4.50	2.44	0.91
	Partial SMR	3.00	1.37	0.78	0.63	0.57	0.56
β_x	Lee-Carter	20.19	13.76	8.61	4.79	2.63	1.32
	Partial SMR	0.27	0.23	0.15	0.10	0.07	0.06

In addition to the MAPE between the predicted and observed mortality rates, we also compute the mean squared error (MSE) of the parameters' estimate (α_x and β_x) for the Lee-Carter model. To a clear understanding, we use the case of Taiwanese men (5-age groups) as a demonstration and the results are shown in Table 6. The proposed partial SMR does provide smaller MSE's for both parameters α_x and β_x, and the MSE's of the partial SMR do not change much for different population sizes. On the other hand, the MSE's of Lee-Carter model are obviously influenced by the population sizes.

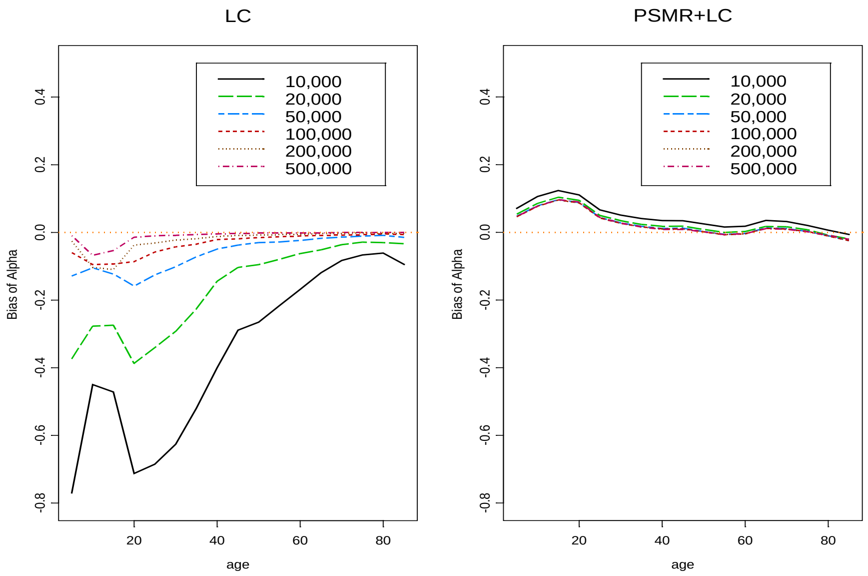

Figure 7. α_x Estimates Bias for Different Models

We can further examine why the partial SMR has smaller MSE for parameters α_x and β_x. Figure 7 shows the bias of α_x estimate for various population sizes in the case of Taiwanese men (5-age groups). Apparently the partial SMR has significantly smaller bias, especially for smaller population sizes, such as 10,000 and 20,000, and this probably can explain why the Lee-Carter model has larger MAPE's for smaller population sizes (Tables 3 and 4). The bias of α_x estimate for the partial SMR do not change much for different population sizes, while those for the Lee-Carter model decrease as population sizes increase. Similarly, the variance of α_x estimate for the partial SMR is very small, unlike those for the Lee-Carter model (Figure 1).

The variance of β_x estimate for the partial SMR is also very small, but the bias of

β_x estimate is not the same. Again, we use the case of Taiwanese men (5-age groups) as an example. Figure 8 shows the bias of β_x estimates with and without plugging the partial SMR smoothing in the cases of population size 10,000 and 20,000. In general, the estimates after the partial SMR have smaller bias, and the largest improvement is at the younger ages. This result matches to those in Figure 5 (life tables of Pen-Hu county) since the mortality rates and number of deaths around ages 5~15 are the smallest. However, the partial SMR still has larger bias at the older ages, although the differences are not significant.

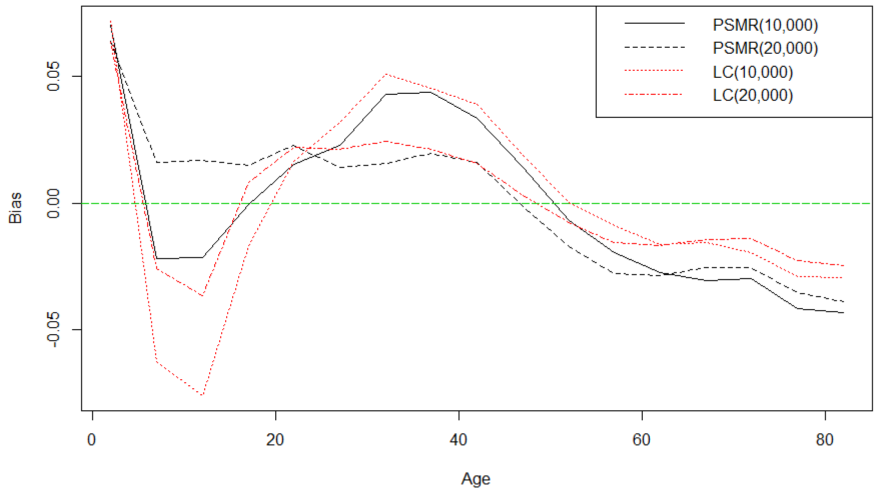

Figure 8. Bias of Parameter β_x before and after the Partial SMR

5. Conclusion and Discussions

Longevity is a popular issue in the 21^{st} century and the study of mortality rates is one of the main research topics in recent years. In addition to the problem of data quality, small population is another main difficulty in exploring mortality models. Although small area estimation is not new, the development of dealing with small population in applying mortality models is quite new. The coherent Lee-Carter model (Li and Lee, 2004) is one of the famous examples for modifying the Lee-Carter model in a sense by increasing the population size. Past studies showed that increasing the population size (or borrowing information from reference populations) does improve the fitting and forecasting of mortality rates (e.g., Booth et al., 2006; Jarner and Kryger, 2011).

The idea of increasing the population size/referencing other populations for modeling mortality rates are not unique in insurance. Credibility and mortality graduation are two other famous applications adapting the same notion, which means that the methodology of graduation/credibility can be used for improving the fitting of mortality models in the case of small populations. Thus, we modify the partial SMR (standard mortality ratio) for graduating mortality rates of small populations (Lee, 2003). We use it to reduce the fluctuations of mortality rates and the model fitting of mortality models.

The small population size can cause difficulties in applying mortality models. For example, the popular Lee-Carter model (Lee and Carter, 1992) has biased estimates for age-related parameters α_x and β_x in the case of small populations. According to our simulation, the bias is especially noticeable when the population size is 200,000 or less. The bias problem is almost the same using different estimation methods, either singular value decomposition (SVD), approximation method, or Poisson approach (Brouhns et al., 2002). If we treat the (20-year) aggregation of historical data of small population as the reference population, then applying the partial SMR to the mortality rates has smaller MAPE's than the Lee-Carter model.

The proposed partial SMR can be applied to any mortality models, such as Gompertz's law and age-period-cohort model, and is not restricted to the Lee-Carter model. We can first apply the partial SMR, with aggregation of 10~20 years of data as the reference population, and then apply the graduated values to mortality models. This might reduce the bias in the parameter estimation but the improvement may not be the same as that of the Lee-Carter model. For example, the bias of parameters' estimates of CBD model (Cairns Blake-Dowd, 2006) are not as significant as those of the Lee-Carter model.

On the other hand, as shown in Figure 6, the partial SMR can reduce the bias of β_x estimates but the improvement rate is not as much as that with respect to MAPE. It is likely that two-stage estimation, i.e., graduation first and then Lee-Carter model, discounts the improvement. We will continue exploring possible modifications for including the partial SMR into the mortality models. Alternatively, if our goal is mortality projection, maybe we can apply simulation methods (such as block bootstrap) directly to the mortality rates after the partial SMR, without going to the stage of model estimation.

Of course, there are possibilities for modifying the partial SMR approach. For example, intuitively, the weight should include the exposure (not restricted to the numbers of deaths), like the Kernel graduation (Gavin et al., 1993). Also, the information from reference population can be SMR of certain age groups, instead of all ages. Bayesian approach is another possibility of increasing the sample size, such as the Bayesian modification of Lee-Carter model by Wiśniowski et al. (2015). We can adapt the Bayesian notion in adjusting the weights of small and reference populations.

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Appendix A. Simulation Errors of 3 Mortality Scenarios
(Population size = 200,000)

a		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Constant	Raw	15.0	14.3	13.6	13.1	12.7	12.3	11.9	11.5	11.3	10.9
	Whittaker	8.5	8.3	8.0	7.7	7.4	7.3	7.0	6.9	6.8	6.6
	PSMR	2.5	2.4	2.2	2.2	2.1	2.0	2.0	1.9	1.9	1.8
Increasing	Raw	14.9	17.2	22.2	27.8	35.5	44.0	54.7	68.3	90.0	132.9
	Whittaker	8.4	10.4	14.6	19.4	25.8	32.8	42.0	53.5	72.5	112.8
	PSMR	2.4	6.2	12.5	19.9	28.5	38.2	50.1	65.6	89.4	138.6
V-shape	Raw	14.9	16.7	21.6	27.4	34.4	42.9	52.9	65.8	85.2	125.0
	Whittaker	8.5	10.5	14.9	20.5	26.9	34.5	43.4	54.9	72.2	109.5
	PSMR	2.4	6.2	12.4	19.5	27.5	36.5	47.0	60.5	80.3	121.0