The Advantage of Second Guesser in a Two-person Game

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Abstract

A guessing game between two persons, say, A and B, is of interest. The rule is as following: A and B each put a certain amount of money into a box, without letting the other knows. The goal is to guess the total money in the box, and the one with the correct guess wins the game and all the money. A guesses first, and then B. We are interested in the advantage of B over A, including the case when A can guess more than once. We also use simulation to check with the results derived.

KEY WORDS: Two-person game, Zero-sum game, Optimal strategies, Conditional probability, Simulations

1. INTRODUCTION

A guessing game, which is often played in our childhood, is of interest. The game is played between two persons, namely A and B. First, A and B must put \$$x$ and \$$y$, respectively, in a box without letting the other knows, where $x, y \in \{1, 2, \ldots, k\}$ and k is a fixed and known number. The goal is to guess the sum of x and y, and the one with the correct guess takes away all the money in the box. Let A guess first, say z. And then B guesses, w ($w \neq z$), with extra information from the number that A guesses. If neither A nor B has the correct guess, the game ends. The following is the list of notation used in this report:

	A	B
Bet	x	y
Guess	z	w

Also, define (A wins) and (B wins) as the events that A and B win the game, and define (A gains) and (B gains) as the money that A and B gain from the game. Further, we assume that it is equally likely for A and B to bet any amount of money, i.e. x and y take value from $1, 2, \cdots, k$ each with probability $1/k$. Based on this assumption, it is natural for A to guess over the range $x+1, x+2, \ldots, x+k$, i.e. $x+1 \leq z \leq x+k$. Similarly, it is natural to assume that $y+1 \leq w \leq y+k$. This kind of assumption on z and w will be defined as *rational guess*, and we only deal with rational guess in this report. The cases when $z \leq x$ and $z \geq x+k+1$, which might misguide the decision of the second guesser, are not considered in the current study.

Given z, the range of w can be further reduced. For example, if $z = 5(\leq k)$, then the bet x must satisfy $1 \leq x \leq 4$ and so $y+1 \leq w \leq y+4$. In general, given z, the possible values of x can be reduced to the interval:

$$\max\{1, z-k\} \leq x \leq min\{k, z-1\}. \tag{1}$$

In the following, the case when both z and w are uniform is considered in Section 2. Section 3 covers the discussion of the case when only z is uniform and the case when only w is uniform. Variations of this two-person guessing game are discussed in Section 4.

2. BOTH z AND w ARE UNIFORM

Since x and y are uniform, it is natural for A to guess every possible combination of $X+Y$ with equal probability, or

$$P(Z = z \mid X = x) = \frac{1}{k}, \qquad z = x+1, x+2, \ldots, x+k.$$

Then the winning probability of A can be computed directly as:

$$P(A\,\text{wins}) = \sum_{y=1}^{k} P(Y=y)P(A\,\text{wins} \mid Y=y) = \frac{1}{k}. \tag{2}$$

Similarly, the expected gain of A is

$$E(A\,\text{gains}) = \sum_{y=1}^{k} y \times P(Y=y)P(A\,\text{wins} \mid Y=y) = \frac{k+1}{2k}. \tag{3}$$

It should be noted that the expected gain of A is just the winning probability of A times the expected bet of B.

The calculation of $P(B\,\text{wins})$ and $E(B\,\text{gains})$ is more complicated, and can be derived via dividing the discussion into two cases: $2 \le z \le k$ and $k+1 \le z \le 2k$. Readers who are interested in the derivation, can see Lu and Yue (1996) for details. Following the result in Lu and Yue, the conditional probability of W, given $Y=y$ and $Z=z$, is

$$P(W=w \mid Y=y, Z=z) \tag{4}$$
$$= \begin{cases} \frac{1}{z-2}, & \text{if } 3 \le z \le k \quad 1 \le y \le z-1 \\ \frac{1}{z-1}, & \text{if } 3 \le z \le k \quad z \le y \le k \\ \frac{1}{2k-z+1}, & \text{if } k+1 \le z \le 2k-1 \quad 1 \le y \le 2k-z+1 \\ \frac{1}{2k-z}, & \text{if } k+1 \le z \le 2k-1 \quad z-k \le y \le k. \end{cases}$$

So, the winning probability of B is

$$P(B\,\text{wins}) = \frac{2k^2-k-2}{k^3}. \tag{5}$$

and the expected gain of B is

$$E(B\,\text{gains}) = \frac{2k^3+k^2-3k-2}{2k^3}. \tag{6}$$

Compare (2) to (5), we can see that the winning probability of B is twice as much as that of A. Similarly, compare (3) to (6), the same pattern also occurs on the expected gain. Thus, the advantage of B over A is clear.

As a check, plugging into $k = 2$, we have $P(B\,\text{wins}) = 1/2$ and $E(B\,\text{gains}) = 3/4$, which matches to our previous result.

Theorem 1. *If x, y, z, w are uniform over possible values, then*

$$P(A\,\text{wins}) = \frac{1}{k} \quad \& \quad P(B\,\text{wins}) = \frac{2k^2 - k - 2}{k^3},$$

$$E(A\,\text{gains}) = \frac{k+1}{2k} \quad \& \quad E(B\,\text{gains}) = \frac{2k^3 + k^2 - 3k - 2}{2k^3}.$$

Here we use simulation to verify the above results. Table 1 lists the simulation of $k = 2, 3, 6, 10, 100$ and each with $10,000$ replications. The top numbers in each cell are the simulated values and the numbers inside the parenthesis are the theoretical values. The simulated values are all closed to the theoretical values in every case.

Table 1 Simulation for various k

	k=2	k=3	k=6	k=10	k=100
P(A wins)	0.4974 (0.5)	0.3361 (0.3333)	0.1710 (0.1667)	0.1014 (0.1)	0.0096 (0.01)
P(B wins)	0.5026 (0.5)	0.4815 (0.4815)	0.2958 (0.2963)	0.1867 (0.188)	0.0163 (0.0199)
E(A gains)	0.7411 (0.75)	0.6712 (0.6667)	0.6058 (0.5833)	0.5454 (0.55)	0.5529 (0.505)
E(B gains)	0.7559 (0.75)	0.9423 (0.963)	1.0346 (1.037)	0.9957 (1.034)	0.9208 (1.0048)

3. ONE OF z AND w IS UNIFORM

Suppose only z is uniform, we are interested in the choices of w which can increase the advantage of B over A. Similarly, if only w is uniform then we want to know the choices which can reduce the advantage of B over A. We will discuss the case when z is uniform first.

(i) If only z is uniform:

From Section 2, we found that the extra information of B from A doubles the expected gain (and the winning probability as well), comparing to that of A. Suppose now we can relax the assumption that w is uniform. The goal is to find the best strategy of w, which maximizes the difference between $P(B\,\text{wins})$ and $P(A\,\text{wins})$, and the difference between $E(B\,\text{gains})$ and $E(A\,\text{gains})$.

Note that $P(A\,\text{wins}) = 1/k$ is independent of w, provided that x and y are uniform. (Actually, only y is uniform is required.) For B, if w is independent of z, then the winning probability of B is $1/k$, same as A. But when z is rational and the choice of w depends on z, the winning probability of B shall be larger than that of A. As we can see from last section, when x, y, z, w are all uniform, $P(B\,\text{wins})$ is about twice of $P(A\,\text{wins})$. In general, when z is uniform, B has larger winning chance than A no matter how B guess as long as w is rational.

Lemma 1. *If y is uniform, then*

$$P(A\,wins) = \frac{1}{k}.$$

If, addition, z is uniform,

$$P(B\,\text{wins}) = P(A\,\text{wins}).$$

The advantage of B over A is clear, but it is interesting to see whether $P(B\,\text{wins})$ can be increased by relaxing the constraint of w. However, it can be shown that

$$P(B\,\text{wins}) = \frac{2k^2 - k - 2}{k^3}$$

if x, y, z are uniform and w is rational. In other words, the winning probability of B cannot be increased by changing the guessing rule of w.

On the other hand, although it is impossible to increase $P(B\,\text{wins})$, we can increase $E(B\,\text{gains})$ by choosing the appropriate w's: Let B guess the largest possible value of x, since z is uniform and $\max\{1, z-k\} \leq x \leq \max\{k, z-1\}$ (from (1)). Thus, the maximum payoff of B is attained when B guesses $x = \max\{k, z-1\}$. In this case,

$$E(B\,\text{gains}) = \frac{3k-1}{2k}$$

and

$$\max\{E(B\,\text{gains}) - E(A\,\text{gains})\} = \frac{2k-2}{2k} \approx 1.$$

Therefore, the advantage of second guesser is even larger when the guess w is chosen to be non-uniform. And the expected gains of B is three times as large as that of A. This can be explained as the adding the condition of relaxing the uniform constraint of w.

(ii) If only w is uniform:

From case (i), the relax of uniform constraint of w has increased about the expected gain of B by $1/2$, or the amount of the expected gain of A when the bets and guesses are all uniform. Thus, it is natural to expect the relax of uniform constraint of z would offset the extra information from z, if w is uniform.

Suppose w is uniform. We want to see if this can narrow the gaps of $\{P(B\,\text{wins}) - P(A\,\text{wins})\}$ and $\{E(B\,\text{gains}) - E(A\,\text{gains})\}$. Since B has more information, intuitively, we still would expect that B has larger winning probability if z is rational and w is uniform.

Lemma 2. *If y and w are uniform, then*

$$P(A\,\textit{wins}) \leq P(B\,\textit{wins})$$

and "=" holds if $z = k+1$ for all x.

Thus, from the above lemma, it is trivial that $\max\{P(B\,\text{wins}) - P(A\,\text{wins})\} = 0$ when $z = k+1$, and A cannot do any better than this.

Similar to lemma 2, we can show that $E(B\,\text{gains}) = E(A\,\text{gains})$ when $z = k+1$, and x and y are uniform. But is it possible that $E(A\,\text{gains}) > E(B\,\text{gains})$? The following example shows one of the possibilities:

Example. Suppose $k = 4$ and x, y, w are all uniform over possible values. Let $z = 5 = k+1$ when $x = 1, 3, 4$ and $z = 6$ when $x = 2$. Then

$$E(A\,\text{gains}) = (\frac{1}{4})^2(4+4+2+1) = \frac{33}{48};$$

while

$$E(B\,\text{gains}) = (\frac{1}{4})^2(1+\frac{8}{3}+3+4) = \frac{32}{48}.$$

So, $E(A\,\text{gains}) - E(B\,\text{gains}) = 1/48 > 0$ and it is possible that $E(A\,\text{gains}) > E(B\,\text{gains})$ when z is not uniform.

In general, the maximum of $E(A\,\text{gains}) - E(B\,\text{gains})$ can be achieved by looking at the possible outcomes when x is given. The following lemmas show the optimal strategy of A to obtain the maximum of $E(A\,\text{gains}) - E(B\,\text{gains})$:

Lemma 3. *If x, y, w are all uniform, then taking the guess $z \leq k$ is never optimal for A.*

Theorem 2. *If x, y, w are all uniform, then the maximum of $E(A\,\text{gains}) - E(B\,\text{gains})$ is achieved when A takes the following strategies:*

(a) Given x, choose the smallest $i (z = k+i)$ satisfying

$$\frac{k}{(k-i)(k-i+1)} \cdot \frac{x}{k} \geq \frac{1}{k}, \tag{7}$$

where $1 \leq i \leq x-1$.

(b) If no $i \geq 1$ satisfies (7), take $z = x+k$.

Proof: Since the proof is straightforward, it is omitted.

To demonstrate the result of Theorem 2, let's consider $k = 4$ and $x = 2$. The smallest number satisfying (7) should be $i = 2$, which explains the choices of $z = 6$ when $k = 4$ and $x = 2$ in the above example. Also, from (7) in lemma 2, the range of the optimal bet z should satisfy

$$k + 1 \leq z \leq k + [\frac{3 - \sqrt{5}}{2} k],$$

where $[k]$ is the biggest number smaller than k. Both $x = 1$ and $x = k$ imply that $z = k+1$, while $x = [\frac{3-\sqrt{5}}{2} k]$ gives the maximum value of $z = k+[\frac{3-\sqrt{5}}{2} k]$.

From Theorem 2, although it is possible that $E(B\,\text{gains}) < E(A\,\text{gains})$, the difference between $E(A\,\text{gains})$ and $E(B\,\text{gains})$ is very small. It can be shown that this difference is of the order $(1/k)$, and will converge to 0 as $k \to \infty$. In other words, the expected gains of A and B is about the same when w is uniform. It can be interpreted as that the extra information given to the second guesser B, is equivalent to the condition that z can be non-uniform.

Theorem 3. *If x, y, w are all uniform, and z follows the strategy in (7), then*

$$P(A\,\text{wins}) \approx P(B\,\text{wins}) \approx \frac{1}{k}$$

and

$$E(A\,\text{gains}) \approx E(B\,\text{gains}) \approx 1.$$

4. SOME VARIATIONS

From (2),(3),(5), and (6), the winning probability of B is about twice as much as that of A, when the bets and guesses are all uniform. The comparison of expected gains between A and B is similar. It would be fair to double the expected gain of A to offset the disadvantage of A. On the other hand, another seemingly fair alternative is to let A guess twice and B just once. Of course, under this setting, the winning probability and the expected gain of A will be doubled, but B will have more information, too. Intuitively, the winning probability and the expected gain of B should still be larger than those of A. The following simulations confirm this idea:

Table 2 Simulations of cases when A guesses twice and B once

	k=3	k=6	k=10	k=20	k=100
P(A wins)	0.6688	0.3371	0.2006	0.0998	0.0202
	(0.0047)	(0.0047)	(0.0040)	(0.0030)	(0.0014)
P(B wins)	0.3312	0.3895	0.2648	0.1431	0.0293
	(0.0047)	(0.0049)	(0.0044)	(0.0035)	(0.0017)
E(A gains)	1.3404	1.1772	1.1017	1.0508	1.0218
	(0.0094)	(0.0165)	(0.0220)	(0.0315)	(0.0707)
E(B gains)	0.6598	1.3789	1.4366	1.4770	1.4895
	(0.0094)	(0.0172)	(0.0242)	(0.0368)	(0.0859)

From Table 2, we can see that the advantage of B over A is about 3/2 times, comparing to 2 times when A only guesses once. Simulations for the case when A guesses 3 times and B once shows similar result, and the advantage of B over A is about 4/3 times, providing that k is large enough. To explain this pattern, we can treat each guess and each guess from the previous guesser as the same level of information. Therefore, no matter how many times that A guesses, B always has one more piece of information than A. So, as long as the bets and guesses are uniform, and k is large enough, B would have larger winning probability and expected gain. In general, if A guesses $m(\ll k)$ times uniformly and B guesses once, then our think that

$$P(A\,\text{wins}) \approx \frac{m}{k},$$

while

$$P(B\,\text{wins}) \approx \frac{m+1}{k}.$$

And asymptotically, $P(A\,\text{wins}) \approx P(B\,\text{wins})$. Similarly, we conjecture that

$$E(A\,\text{gains}) \approx \frac{m}{2}$$

and

$$E(B\,\text{gains}) \approx \frac{m+1}{2}.$$

The study of the case when A can guess more than once will be discussed more.

Another possible variation is to guess the number of coins (or bills) used by A and B, in addition to the sum of x and y. Then A must guess the number of coins used, as well as the sum of x and y. This means that B might have more advantage over A since there is more information available, provided that A won't give false information. For example, let us consider \$1, \$5, \$10, and \$50 coins. (These are the only NT dollar coins which are currently in circulation.) The following is example showing that the winning probability is actually increased by adding the number of coins used.

Example. Suppose $k = 10$. Assume that A guesses $z = 12$ and the number of coins used is 4. Combining these two guesses, we know that the only possibility is two \$1 coins and two 2 \$5 coins. Further, $z = 12$ and $k = 10$ implies that $2 \le x \le 10$. However, the only possible choices of x shall be $2, 5, 6, 7, 10$, since other values would suggest there are more than 2 \$1 coins. So, the number of possible values of x is reduced to 5, from the original 9.

The general case of adding the number of coins used is more complicated. For B, the possible values of x ($\max\{0, z-k\} \le x \le \max\{k, z-1\}$) and the possible coin combinations of z, shall be considered at the same time, before B makes the guesses.

5. COMMENTS

In this study, the two-person game discussed is of the type zero-sum. It is natural to think that the one with more information has better odd to win the game, and higher expected gain. Therefore, when x, y, z, w are all uniform, the second guesser B has more advantage, both in the winning probability and expected gain. This advantage is even larger, when the restriction of w is relaxed. On the other hand, the effect of relaxing the restriction of z offset the information from z, and makes A and B have about the same winning probability and expected gain. From this study, when x and y are uniform, it seems that the information from z and relaxing the restriction on the guesses (z and w) have the same effect.

The cases discusses in this report are based on the assumption that both x and y are uniform. In reality, this is very unlikely. It would be interesting to explore the cases when the distributions of x and y are given, and find out what distribution gives the largest advantage for B. Also, in this study, since we assume guesses are rational, interested readers could discuss the possible outcomes for changing any of the assumptions. In particular, suppose A has the option to give the false information (i.e. z can be not rational). Then B must decide whether to take or discard the information from A. Furthermore, when A guesses two or more than two times, and A may give false information, the advantage of B may be reduced and the information may even be misleading. Then, for B, it might be better to guess uniformly according to y, regardless the guesses from A. In this case, B is more like the first guesser, instead of the second guesser.

REFERENCES

1. Coe, P. R. and Butterworth, W. (1995), "Optimal Stopping in 'The Showcase Showdown'", *The American Statistician*, 49, 271-275.

2. Lu, M. C. and Yue, J. C. (1995) "The Advantage of Second Guesser in a Two-person Game", *Technical report, Department of Statistics, National Chengchi University.*

3. Morgan, J. P., Chaganty, N. R., Dahiya, R. C., and Doviak, M. J. (1991), "Let's Make a Deal: The Player's Dilemma", *The American Statistician*, 45, 284-287.

4. Steele, J. M. and Zidek, J. (1980), "Optimal Strategies for Second Guessers", *Journal of the American Statistical Association*, 75, p. 596-601.